Semigroup Forum (2022) 105:345–349
https://doi.org/10.1007/s00233-022-10305-2

SHORT NOTE

Identities of the stylic monoid

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Received: 16 June 2022 / Accepted: 12 July 2022 / Published online: 1 August 2022

Abstract
We observe that for each $n \geq 2$, the identities of the stylic monoid with n generators coincide with the identities of n-generated monoids from other distinguished series of $\mathscr{J}$-trivial monoids studied in the literature, e.g., Catalan monoids and Kiselman monoids. This solves the Finite Basis Problem for stylic monoids.

Keywords $\mathscr{J}$-trivial monoid · Stylic monoid · Kiselman monoid · Catalan monoid · Finite Basis Problem

Mathematics Subject Classification 20M05 · 20M07

A *monoid identity* is a pair of *words*, i.e., elements of the free monoid X^* over an alphabet X, written as a formal equality. An identity $w = w'$ with $w, w' \in X^*$ is said to *hold in a monoid* M if $w\varphi = w'\varphi$ for each homomorphism $\varphi\colon X^* \to M$; alternatively, we say that the monoid *satisfies* $w = w'$ or that $w = w'$ is an identity of M.

Given any set Δ of monoid identities, we say that an identity $w = w'$ *follows* from Δ if every monoid satisfying all identities of Δ satisfies the identity $w = w'$ as well. Birkhoff's completeness theorem of equational logic (see [4, Theorem 14.17]) shows that this notion (which we have given a semantic definition) is captured by a transparent set of inference rules. The syntactic viewpoint is often useful but is not utilized in this note.

Given a monoid M, a set Δ of its identities is said to be an *identity basis for* M if every identity holding in M follows from Δ. The *Finite Basis Problem* for a monoid M is the question of whether or not M admits a finite identity basis.

A monoid M is said to be $\mathscr{J}$*-trivial* if every principal ideal of M has a unique generator, that is, $MaM = MbM$ implies $a = b$ for all $a, b \in M$. Finite $\mathscr{J}$-trivial

Communicated by Dominique Perrin.

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monoids attract much attention because of their distinguished role in algebraic theory of regular languages [15, 17, 18] and representation theory [9].

Several series of finite $\mathscr{J}$-trivial monoids parameterized by positive integers appear in the literature, including Straubing monoids [20, 21], Catalan monoids [19], double Catalan monoids [16], Kiselman monoids [11, 13, 14], and gossip monoids [7, 10, 12]. These monoids arise due to completely unrelated reasons and consist of elements of a very different nature. Surprisingly, studying identities of the listed monoids has revealed that the n-th monoids in each series satisfy exactly the same identities! This was first observed by the author [21] for Straubing and Catalan monoids. Then, in [3] the result was extended to Kiselman monoids, and more generally, to a wide spectrum of Hecke–Kiselman monoids from [11]. Marianne Johnson and Peter Fenner [12] have added double Catalan, gossip, and one-directional gossip monoids to the list.

Recently, a new family of finite $\mathscr{J}$-trivial monoids, coined *stylic* monoids, has been introduced by Antoine Abram and Christophe Reutenauer [1], with motivation coming from combinatorics of Young tableaux. It is quite natural to ask whether the above phenomenon extends to this new family, i.e., whether the n-th stylic monoid again satisfies the same identities as the n-th monoids in each aforementioned series. The present note aims to answers this question in the affirmative. When it was submitted and its preprint version [22] appeared on arXiv, Duarte Ribeiro informed the author that he and Thomas Aird obtained the same result independently and contemporaneously, albeit with a somewhat more complicated proof, see [2].

The combinatorial definition of stylic monoids can be found in [1], but here we only need their presentation via generators and relations established in [1, Theorem 8.1(ii)]. Thus, let the stylic monoid Styl_n be the monoid generated by $a_1, a_2, \dots, a_n$ subject to the relations

$$a_i^2 = a_i \qquad \text{for each } i = 1, \dots, n; \qquad (1)$$

$$a_j a_i a_k = a_j a_k a_i \qquad \text{if } 1 \le i < j < k \le n; \qquad (2)$$

$$a_i a_k a_j = a_k a_i a_j \qquad \text{if } 1 \le i < j < k \le n; \qquad (3)$$

$$a_j a_i a_i = a_i a_j a_i \qquad \text{if } 1 \le i < j \le n; \qquad (4)$$

$$a_j a_j a_i = a_j a_i a_j \qquad \text{if } 1 \le i < j \le n. \qquad (5)$$

We want to relate Styl_n to two other monoids with same generating set. The Kiselman monoid Kis_n, defined by Christer Kiselman [13] for $n = 3$ and by Olexandr Ganyushkin and Volodymyr Mazorchuk (unpublished) for an arbitrary $n \ge 2$, is generated by $a_1, a_2, \dots, a_n$ subject to the relations

$$a_i^2 = a_i \qquad \text{for each } i = 1, \dots, n; \qquad (6)$$

$$a_i a_j a_i = a_j a_i a_j = a_j a_i \qquad \text{if } 1 \le i < j \le n. \qquad (7)$$

Catalan monoids were defined by Andrew Solomon [19] as monoids of certain transformations on directed paths, but again, we only need their presentation via generators and relations from [19, Section 9], see also [11] for a short argument. So, for the purpose of this note, let Cat_n stand for the monoid generated by $a_1, a_2, \dots, a_n$ subject to the relations

$$a_i^2 = a_i \qquad \text{for each } i = 1, \dots, n; \tag{8}$$
$$a_i a_k = a_k a_i \qquad \text{if } |i-k| \geq 2,\ i,k = 1, \dots, n; \tag{9}$$
$$a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1} = a_{i+1} a_i \qquad \text{for each } i = 1, \dots, n-1. \tag{10}$$

Lemma 1 *The relations* (2)–(5) *hold in the monoid* Cat$_n$.

Proof If $i, j, k \in \{1, 2, \dots, n\}$ are such that $i < j < k$, then $k - i \geq 2$ whence $a_j a_i a_k \overset{(9)}{=} a_j a_k a_i$ and $a_i a_k a_j \overset{(9)}{=} a_k a_i a_j$ in Cat$_n$. Thus, (2) and (3) hold in Cat$_n$.

If $1 \leq i < j \leq n$ and $j - i \geq 2$, we can apply (9) in a similar way: $a_j a_i a_i \overset{(9)}{=} a_i a_j a_i$ and $a_j a_j a_i \overset{(9)}{=} a_j a_i a_j$. Therefore, to prove that (4) and (5) hold in Cat$_n$, it remains to consider the case $j = i + 1$. In this case, we can deduce in Cat$_n$ the following:

$$a_{i+1} a_i a_i \overset{(8)}{=} a_{i+1} a_i \overset{(10)}{=} a_{i+1} a_i a_{i+1} \quad \text{and} \quad a_{i+1} a_{i+i} a_i \overset{(8)}{=} a_{i+1} a_i \overset{(10)}{=} a_i a_{i+1} a_i.$$

Hence (4) and (5) also hold in Cat$_n$ for all i, j with $1 \leq i < j \leq n$. □

Lemma 2 *The relations* (7) *hold in the monoid* Styl$_n$.

Proof For any $i, j \in \{1, 2, \dots, n\}$ with $i < j$, we can deduce in Styl$_n$ the following:

$$a_i a_j a_i \overset{(4)}{=} a_j a_i a_i \overset{(1)}{=} a_j a_i \quad \text{and} \quad a_j a_i a_j \overset{(5)}{=} a_j a_j a_i \overset{(1)}{=} a_j a_i.$$

Hence the relation $a_i a_j a_i = a_j a_i a_j = a_j a_i$ holds in Styl$_n$. □

Theorem 1 *For each $n \geq 2$, the monoid* Cat$_n$ *is a homomorphic image of* Styl$_n$*, and the monoid* Styl$_n$ *is a homomorphic image of* Kis$_n$.

Proof We invoke Dyck's Theorem (see, e.g., [8, Theorem III.8.3]). Specialized in the case of monoids, it says that if M is a monoid generated by a set A subject to relations R and N is a monoid generated by A and such that all the relations R hold in N, then N is a homomorphic image of M. In view of this fact, Lemma 1 implies that Cat$_n$ is a homomorphic image of Styl$_n$, while Lemma 2 ensures that Styl$_n$ is a homomorphic image of Kis$_n$. □

Corollary 1 *For each $n \geq 2$, the monoids* Cat$_n$, Styl$_n$*, and* Kis$_n$ *satisfy the same identities.*

Proof Clearly, if a monoid M satisfies an identity, then so does every homomorphic image of M. Therefore, Theorem 1 implies that every identity holding in Kis$_n$ holds in Styl$_n$, and every identity holding in Styl$_n$ holds in Cat$_n$. However, it follows from [3, Theorem 8] that for each $n \geq 2$, the monoids Cat$_n$ and Kis$_n$ satisfy the same identities. Hence the same identities hold in the 'intermediate' monoid Styl$_n$ as well. □

Since the identities of the monoids Cat_n and Kis_n have been characterized in [3, 21], Corollary 1 leads to an efficient combinatorial description of the identities of stylic monoids. The description involves the notion of a scattered subword. Recall that a product $x_1 \cdots x_k$ of elements from an alphabet X is said to be a *scattered subword of length* k in a word $v \in X^*$ if there exist words $v_0, v_1, \ldots, v_{k-1}, v_k \in X^*$ such that $v = v_0 x_1 v_1 \cdots v_{k-1} x_k v_k$; in other terms, $x_1 \cdots x_k$ is a subsequence of v. The following is a combination of Corollary 1 with either [21, Proposition 4 and Corollary 2] or [3, Theorem 8].

Corollary 2 *An identity* $w = w'$ *holds in the monoid* Styl_n *if and only if the words* w *and* w' *have the same set of scattered subwords of length at most* n.

As yet another immediate application, we get a solution to the Finite Basis Problem for stylic monoids. It comes from Corollary 2 combined with results by Francine Blanchet-Sadri [5, 6].

Corollary 3 (a) *The identities* $xyxzx = xyzx,\ (xy)^2 = (yx)^2$ *form an identity basis for the monoid* Styl_2.
(b) *The identities* $xyx^2zx = xyxzx,\ xyzx^2tz = xyxzx^2tz,\ zyx^2ztx = zyx^2zxtx,\ (xy)^3 = (yx)^3$ *form an identity basis for the monoid* Styl_3.
(c) *The monoid* Styl_n *with* $n \geq 4$ *admits no finite identity basis.*

Acknowledgements The useful suggestions of the referee are gratefully acknowledged.
The work was supported by the Ministry of Science and Higher Education of the Russian Federation (Ural Mathematical Center project No. 075-02-2022-877)

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