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RESEARCH ARTICLE

On characterization by Gruenberg–Kegel graph of finite simple exceptional groups of Lie type

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In memory of Irina Dmitrievna Suprunenko

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Abstract

The Gruenberg–Kegel graph (or the prime graph) $\Gamma(G)$ of a finite group G is the graph whose vertex set is the set of prime divisors of $|G|$ and in which two distinct vertices r and s are adjacent if and only if there exists an element of order rs in G. A finite group G is called *almost recognizable* (by Gruenberg–Kegel graph) if there is only a finite number of pairwise non-isomorphic finite groups having Gruenberg–Kegel graph as G. If G is not almost recognizable, then it is called *unrecognizable* (by Gruenberg–Kegel graph). Recently Peter J. Cameron and the first author have proved that if a finite group is almost recognizable, then the group is almost simple. Thus, the question of which almost simple groups (in particular, finite simple groups) are almost recognizable is of prime interest. We prove that every finite simple exceptional group of Lie type, which is isomorphic to neither ${}^2B_2(2^{2n+1})$ with $n \geqslant 1$ nor $G_2(3)$ and whose Gruenberg–Kegel

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graph has at least three connected components, is almost recognizable. Moreover, groups ${}^2B_2(2^{2n+1})$, where $n \geqslant 1$, and $G_2(3)$ are unrecognizable.

Keywords Finite group · Simple group · Exceptional group of Lie type · Gruenberg–Kegel graph (prime graph) · Recognition by Gruenberg–Kegel graph

Mathematics Subject Classification 20D60 · 20D06 · 05C99

1 Introduction

Throughout the paper we consider only finite groups and simple graphs, and henceforth the term group means a finite group and the term graph means a simple graph, that is an undirected graph without loops and multiple edges.

Let G be a group. The *spectrum* $\omega(G)$ is the set of all element orders of G. The *prime spectrum* $\pi(G)$ is the set of all primes belonging to $\omega(G)$. A graph $\Gamma(G)$ whose vertex set is $\pi(G)$ and in which two distinct vertices r and s are adjacent if and only if $rs \in \omega(G)$ is called the *Gruenberg–Kegel graph* or the *prime graph* of G.

We say that the group G is

- *recognizable* (by Gruenberg–Kegel graph) if for every group H the equality $\Gamma(H) = \Gamma(G)$ implies that $G \cong H$;
- *k-recognizable* (by Gruenberg–Kegel graph), where k is a positive integer, if there are exactly k pairwise non-isomorphic groups having the same Gruenberg–Kegel graph as G;
- *almost recognizable* (by Gruenberg–Kegel graph) if it is k-recognizable by Gruenberg–Kegel graph for a positive integer k;
- *unrecognizable* (by Gruenberg–Kegel graph) if there are infinitely many pairwise non-isomorphic groups having the same Gruenberg–Kegel graph as G.

Note that groups can be characterized by various numerical sets. For example, if we replace $\Gamma(G)$ in these definitions with $\omega(G)$, then we obtain the corresponding definitions for recognizability by spectrum. Nevertheless, if the characterization set is not specified, we suppose that it is the Grunberg–Kegel graph. It is easy to see that if $\omega(G) = \omega(H)$ for groups G and H, then $\Gamma(G) = \Gamma(H)$, however, the converse implication is not true in general. Consider alternating groups A_5 and A_6. Clearly, $\Gamma(A_5) = \Gamma(A_6)$, where both graphs are empty graphs on three vertices 2, 3, and 5, and, on the other hand, we see that $4 \in \omega(A_6) \setminus \omega(A_5)$.

Recently Cameron and the first author have proved [7] that a group G is almost recognizable if and only if each group H with $\Gamma(G) = \Gamma(H)$ is almost simple. Thus, the question of which almost simple groups are almost recognizable is of prime interest. In the same paper [7], a survey of known results on recognition of simple groups has been presented. Note that there are not many completed results at the moment. The situation is much better in the case of the characterization problem by spectrum, where recognition is established for many nonabelian simple groups [14].

In [26], Mazurov conjectured that if a simple group G is not isomorphic to A_6 and $\Gamma(G)$ has at least three connected components, then G is recognizable by its spectrum.

This conjecture was proved in a series of papers, the final result was obtained in [22], where the author proved that groups $E_7(2)$ and $E_7(3)$ are recognizable by Gruenberg–Kegel graph, therefore, these groups are recognizable by spectrum. In the current paper, we refine results related to Mazurov's conjecture. We consider the recognition problem of the simple exceptional groups of Lie type whose Gruenberg–Kegel graphs have at least three connected components.

Connected components of Gruenberg–Kegel graphs of simple groups were described in [21, 36]. A complete result after corrections of mistakes can be found, for example, in [2, Tables 1–3]. If G is a simple group, then $\Gamma(G)$ has at least three connected components if and only if one of the following statements holds:

(1) $G \cong G_2(q)$, where q is a power of 3, or $G \cong {}^2G_2(q)$, where $q = 3^{2n+1} > 3$;
(2) $G \cong {}^2B_2(q) \cong \mathrm{Sz}(q)$, where $q = 2^{2n+1} > 2$;
(3) $G \cong F_4(q)$, where q is even, or $G \cong {}^2F_4(q)$ for $q = 2^{2n+1} > 2$;
(4) $G \cong E_8(q)$;
(5) $G \cong A_1(q) \cong \mathrm{PSL}_2(q)$, where $q > 3$;
(6) $G \cong {}^2D_n(3) \cong P\Omega^-_{2n}(3)$, where $n = 2^m + 1 \geqslant 3$ is a prime;
(7) G is one of the following finite simple groups of Lie type: ${}^2A_5(2) \cong \mathrm{PSU}_6(2)$, $E_7(2)$, $E_7(3)$, $A_2(4) \cong \mathrm{PSL}_3(4)$, ${}^2E_6(2)$;
(8) G is one of the following finite simple sporadic groups: M_{11}, M_{23}, M_{24}, J_3, HiS, Suz, Co_2, Fi_{23}, F_3, F_2, M_{22}, J_1, O′N, LyS, Fi'_{24}, F_1, J_4;
(9) $G \cong A_n$, where $n > 6$ and both n and $n - 2$ are primes.

The main result of this paper is the following statement.

Main Theorem *Every finite simple exceptional group of Lie type, which is isomorphic to neither ${}^2B_2(2^{2n+1})$ with $n \geqslant 1$ nor $G_2(3)$ and whose Gruenberg–Kegel graph has at least three connected components, is almost recognizable by Gruenberg–Kegel graph. Moreover, groups ${}^2B_2(2^{2n+1})$, where $n \geqslant 1$, and $G_2(3)$ are unrecognizable by Gruenberg–Kegel graph.*

In fact, much has been known about the groups in the theorem before in the context of the recognition problem. The groups $E_7(2)$, $E_7(3)$, and ${}^2E_6(2)$ are known to be recognizable (see [22, 23]). The recognizability of simple groups ${}^2G_2(q)$ with $q > 3$ was established in [37].

If G is a nonabelian simple group, then we say that G is *quasirecognizable* by its Gruenberg–Kegel graph if every group H with $\Gamma(G) = \Gamma(H)$ has a unique nonablelian composition factor S and $S \cong G$. In [40], it was proved that each group $G_2(q)$, where $q = 3^n$ and n is odd, is quasirecognizable; however, this result contains an error since $\Gamma(G_2(3)) = \Gamma(\mathrm{PSL}_2(13))$. Quasirecognizability of the simple groups $F_4(q)$ and ${}^2F_4(q)$, where $q > 2$ is even, was proved in [1, 20], respectively. We prove Main Theorem for groups $G_2(q)$, where $q > 3$, $F_4(q)$, and ${}^2F_4(q)$ in Sect. 4. The quasirecognizability of groups ${}^2B_2(q)$ was proved in [40]. We consider the groups ${}^2B_2(q)$ and $G_2(3)$ in Sect. 5.

In [39], Zavarnitsine proved that if $G = E_8(q)$, where $q \equiv 0, \pm 1 \pmod 5$, and H is a group such that $\Gamma(H) = \Gamma(G)$, then $H \cong E_8(u)$ for a prime power $u \equiv 0, \pm 1 \pmod 5$. In Sect. 6, we prove a similar result for groups $E_8(q)$, where $q \equiv \pm 2 \pmod 5$. Therefore, as a corollary, Main Theorem holds for the groups $E_8(q)$.

However, the question of quasirecognizability for the groups $E_8(q)$ is still open (see Sect. 6 for details).

Note that for the remaining simple groups G whose Gruenberg–Kegel graphs have at least three connected components, the recognition problem is still open in the following cases: $G \cong \mathrm{PSL}_2(q)$, where $q > 8$ is even or q is prime and $q \equiv 1 \pmod{12}$; $G \cong A_n$, where $n > 13$ and both n and $n-2$ are primes (for details see the survey in [7]). We suggest that known results on recognition of the groups $\mathrm{PSL}_2(q)$ by Gruenberg–Kegel graph need a careful revision since the authors in their proofs refer to the paper [16] which contains a number of rather serious inaccuracies.

2 Preliminaries

Let n be an integer. Denote by $\pi(n)$ the set of all prime divisors of n. Let π be a set of primes. The largest divisor m of n such that $\pi(m) \subseteq \pi$ is called the *π-part* of n and is denoted by n_π. By π' we denote the set of primes which do not belong to π. If π consists of a unique element p, then we will write n_p and $n_{p'}$ instead of $n_{\{p\}}$ and $n_{\{p\}'}$, respectively.

If n is an integer and r is an odd prime with $(r, n) = 1$, then $e(r, n)$ denotes the multiplicative order of n modulo r. Given an odd integer n, we put $e(2, n) = 1$ if $n \equiv 1 \pmod 4$, and $e(2, n) = 2$ otherwise.

The following lemma is proved in [3], and also in [41].

Lemma 2.1 (Bang–Zsigmondy) *Let q be an integer greater than* 1*. For every positive integer m there exists a prime r with $e(r, q) = m$ besides the cases $q = 2$ and $m = 1$, $q = 3$ and $m = 1$, and $q = 2$ and $m = 6$.*

Fix an integer a with $|a| > 1$. A prime r is said to be a primitive prime divisor of $a^i - 1$ if $e(r, a) = i$. We write $r_i(a)$ to denote some primitive prime divisor of $a^i - 1$ if such a prime exists, and $R_i(a)$ to denote the set of all such divisors.

Lemma 2.2 ([13, Lemma 6]) *Let $q, m,$ and k be positive integers. Then $R_{mk}(q) \subseteq R_m(q^k)$. If, in addition, $(m, k) = 1$, then $R_m(q) \subset R_m(q^k)$.*

Lemma 2.3 *If $r \in R_i(q)$, then $r = ik + 1$, where k is a positive integer.*

Proof This is a consequence of Fermat's little theorem. □

Given a positive integer $i \neq 2$, denote by $k_i(a)$ the product of all primitive prime divisors of $a^i - 1$ with multiplicities counted. If $i = 2$, then put $k_2(a) = k_1(-a)$. It is not difficult to verify that if i is divisible by 4 then $k_i(a) = k_i(-a)$ and if i is odd then $k_i(a) = k_{2i}(-a)$. The following general formula [28] expresses $k_i(a)$, where $i > 2$, in terms of cyclotomic polynomials:

$$k_i(a) = \frac{\Phi_i(a)}{(r, \Phi_{(i)_{r'}}(a))}, \tag{1}$$

where r is the greatest prime divisor of i.

The following assertion is well known and its proof is elementary.

Lemma 2.4 *Suppose that $q > 1$ is an integer. For a positive integer i, an odd prime r divides $q^i - 1$ if and only if $e(r, q)$ divides i.*

The following lemma is a particular case of the well-known Nagell–Ljunggren equation.

Lemma 2.5 ([27]) *Suppose that x, y, and k are positive integers. If $x^2 + x + 1 = y^k$, then either $k = 1$ or $k = 3$, $x = 18$, and $y = 7$.*

The following technical lemma follows from Lemma 2.5, but we give an independent proof. This statement will be needed in the proof of Main Theorem for the group $G_2(q)$.

Lemma 2.6 *Suppose that q is an integer greater than 2 and $\pi(q^2 + \varepsilon q + 1) = \{r\}$, where $\varepsilon \in \{+, -\}$ and r is a prime. Then $r \equiv 1 \pmod 6$. Moreover, if $q \equiv 1 \pmod 8$ then either $r \equiv 1 \pmod 8$ or $r \equiv 3 \pmod 8$.*

Proof By assumption, there exists a positive integer n such that $q^2 + \varepsilon q + 1 = r^n$. Clearly, r is odd. Note that $q^2 + \varepsilon q + 1$ is not divisible by 9, so if $r = 3$, then $n = 1$ and $q \in \{1, 2\}$. If $r \neq 3$, then r divides $k_3(\varepsilon q) = \frac{q^2+\varepsilon q+1}{(q-\varepsilon 1,3)}$ and hence $r \in R_3(\varepsilon q)$. It follows from Lemma 2.3 that $r \equiv 1 \pmod 6$.

Suppose that $q \equiv 1 \pmod 8$. Then either $q^2+\varepsilon q+1 \equiv 1 \pmod 8$ or $q^2+\varepsilon q+1 \equiv 3 \pmod 8$. Therefore, if $r \equiv 5 \pmod 8$ or $r \equiv 7 \pmod 8$, then n is even. On the other hand, $(q-1) < q^2 - q + 1 < q^2$ and $q^2 < q^2 + q + 1 < (q+1)^2$, so n cannot be even. This implies that either $r \equiv 1 \pmod 8$ or $r \equiv 3 \pmod 8$. □

Let G be a finite group. Denote the number of connected components of $\Gamma(G)$ by $s(G)$, and the set of connected components of $\Gamma(G)$ by $\{\pi_i(G) \mid 1 \leqslant i \leqslant s(G)\}$; for a group G of even order, we assume that $2 \in \pi_1(G)$. Denote by $t(G)$ the *independence number* of $\Gamma(G)$, that is the greatest size of a coclique (i. e. induced subgraph with no edges) in $\Gamma(G)$. If $r \in \pi(G)$, then denote by $t(r, G)$ the greatest size of a coclique in $\Gamma(G)$ containing r.

Lemma 2.7 *Let A and B be normal subgroups of a group G such that $A \leqslant B$. If $r, s \in \pi(B/A) \setminus (\pi(A) \cup \pi(G/B))$, then r and s are adjacent in $\Gamma(G)$ if and only if r and s are adjacent in $\Gamma(B/A)$.*

Proof The proof of this lemma is elementary. □

Lemma 2.8 ([32]) *Let G be a finite group with $t(G) \geqslant 3$ and $t(2, G) \geqslant 2$. Then the following statements hold:*

(1) *There exists a nonabelian simple group S such that $S \trianglelefteq \overline{G} = G/K \leqslant \mathrm{Aut}(S)$, where K is the solvable radical of G (i. e., the largest solvable normal subgroup of G).*
(2) *For every coclique ρ of $\Gamma(G)$ of size at least three, at most one prime in ρ divides the product $|K| \cdot |\overline{G}/S|$. In particular, $t(S) \geqslant t(G) - 1$.*
(3) *One of the following two conditions holds:*

(3.1) *$S \cong A_7$ or $L_2(q)$ for some odd q, and $t(S) = t(2, S) = 3$.*

(3.2) *Every prime $p \in \pi(G)$ nonadjacent to 2 in $\Gamma(G)$ does not divide the product $|K| \cdot |\overline{G}/S|$. In particular, $t(2, S) \geqslant t(2, G)$.*

Lemma 2.9 ([29, Lemma 1]) *Let $N \trianglelefteq G$ be an elementary abelian subgroup and $H = G/N$. Define a homomorphism $\phi \colon H \to \mathrm{Aut}(N)$ as follows $n^{\phi(gN)} = n^g$. Then $\Gamma(G) = \Gamma(N \rtimes_\phi H)$.*

Lemma 2.10 ([7, Proposition 3.1]) *Let π be a finite set of primes. The number of pairwise nonisomorphic nonabelian simple groups S with $\pi(S) \subseteq \pi$ is finite, and is at most $O(|\pi|^3)$.*

Let S be a finite simple group of Lie type in characteristic p. Let A be any abelian p-group with an S-action. An element $s \in S$ is said to be *unisingular* on A if s has a (nonzero) fixed point on A. The group S is said to be *unisingular* if every element $s \in S$ acts unisingularly on every finite abelian p-group A with an S-action. Denote by $\mathrm{PSL}_n^\varepsilon(q)$, where $\varepsilon \in \{+, -\}$, the group $\mathrm{PSL}_n(q)$ if $\varepsilon = 1$ and $\mathrm{PSU}_n(q)$ if $\varepsilon = -1$. Similarly, $E_6^\varepsilon(q)$ denotes the simple group $E_6(q)$ if $\varepsilon = 1$ and ${}^2E_6(q)$ if $\varepsilon = -1$.

Lemma 2.11 ([15, Theorem 1.3]) *A finite simple group S of Lie type of characteristic p is unisingular if and only if S is one of the following:*

(i) $\mathrm{PSL}_n^\varepsilon(p)$ *with $\varepsilon \in \{+, -\}$ and n divides $p - \varepsilon 1$;*
(ii) $P\Omega_{2n+1}(p)$, $\mathrm{PSp}_{2n}(p)$ *with p odd;*
(iii) $P\Omega_{2n}^\varepsilon(p)$ *with $\varepsilon \in \{+, -\}$, p odd, and $\varepsilon = (-1)^{n(p-1)/2}$;*
(iv) ${}^2G_2(q)$, $F_4(q)$, ${}^2F_4(q)$, $E_8(q)$ *with q arbitrary;*
(v) $G_2(q)$ *with q odd;*
(vi) $E_6^\varepsilon(p)$ *with $\varepsilon \in \{+, -\}$ and 3 divides $p - \varepsilon 1$;*
(vii) $E_7(p)$ *with p odd.*

Lemma 2.12 ([39, Proposition 2]) *Let $G = {}^3D_4(q)$ act on a nonzero vector space V over a field of characteristic not dividing q (possibly, zero). Then each element of G of order $q^4 - q^2 + 1$ fixes in V a nonzero vector.*

The following lemma is also well known, but we provide its proof for completeness.

Lemma 2.13 *Let G be a group, $g \in G$ an element of order r, and ϕ a nontrivial irreducible representation of G on a nonzero vector space V. If the minimum polynomial degree of $\phi(g)$ equals to r, then g fixes in V a nonzero vector.*

Proof Let $A = \phi(g)$. Since $g^r = 1$, we have $A^r = 1$, and, therefore, the minimal polynomial for A divides the polynomial $x^r - 1$. Since the minimum polynomial degree of A equals to r, we have that the minimum polynomial for A is $x^r - 1$.

By the Cayley–Hamilton theorem, A is a root of its characteristic polynomial. In particular, 1 is an eigenvalue of A, and, therefore, each eigenvector of A which corresponds to the eigenvalue 1, is fixed by $A = \phi(g)$. □

Lemma 2.14 ([31, Theorem 1.1]) *Let G be one of the groups ${}^2B_2(q)$, where $q > 2$, ${}^2G_2(q)$, where $q > 3$, ${}^2F_4(q)$, $G_2(q)$, ${}^3D_4(q)$. Let $g \in G$ an element of prime power order coprime to q. Let ϕ be a nontrivial irreducible representation of G over a field F of characteristic l coprime to q. Then the minimum polynomial degree of $\phi(g)$ equals $|g|$, unless possibly when $G = {}^2F_4(8)$, $l = 3$, $p = 109$ and $\phi(1) < 64692$.*

Lemma 2.15 ([10, Lemma 4]) *Let G be a finite simple group, F a field of characteristic $p > 0$, V an absolute irreducible GF-module, and β a Brauer character of V. If $g \in G$ is an element of prime order distinct from p, then*

$$\dim C_V(g) = (\beta_{\langle g \rangle}, 1_{\langle g \rangle}) = \frac{1}{|g|} \sum_{x \in \langle g \rangle} \beta(x).$$

Lemma 2.16 *Let G be a group with a nontrivial nilpotent normal subgroup K such that G/K has a subgroup H isomorphic to $E_8(q)$, where q is a prime power. Then $R_{24}(q) \subset \pi_1(G)$.*

Proof Note that $R_{24}(q) \subset \pi(H)$.

Let $q = p^l$, where p is a prime. Suppose that K is a p-group. Factoring G and K by K', we can assume that K is abelian. According to [2, Table 3], p lies in $\pi_1(G)$. By Lemma 2.11, H is unisingular and hence p is adjacent to each element of $\pi(H)$. Therefore, $R_{24}(q) \subset \pi_1(G)$.

Take any $r \in \pi(K)$. Since Sylow 2-subgroups of H are non-cyclic and not generalized quaternion, we infer that either $r = 2$ or 2 and r are adjacent in $\Gamma(G)$ (see, for example, [11, Theorem 10.3.1]). Therefore, $r \in \pi_1(G)$.

Suppose that there exists $r \in \pi(K) \cap R_{24}(q)$. Since $\pi(K)$ is a clique in $\Gamma(G)$ and $r \in \pi_1(G)$, we get that $R_{24}(q) \subset \pi_1(G)$.

If K is not a p-group and $\pi(K) \cap R_{24}(q) = \varnothing$, then we can choose $r \in \pi(K) \setminus (\{p\} \cup R_{24}(q))$. According to [24, Table 5.1], H has a subgroup isomorphic to ${}^3D_4(q^2)$. By Lemma 2.12, r is adjacent in $\Gamma(G)$ to any prime from $R_{24}(q)$. This implies that $R_{24}(q) \subset \pi_1(G)$. □

3 The Grunberg–Kegel graphs of some exceptional groups of Lie type

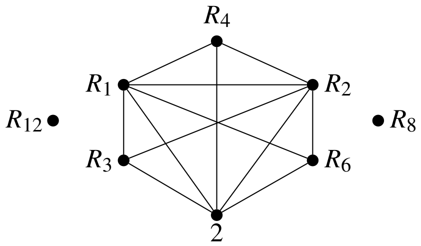
A criterion for the adjacency of vertices in the Gruenberg–Kegel graph for all finite nonabelian simple groups was obtained in [34]. Based on this paper and [35], in this section, we collect the necessary information for exceptional groups of Lie type from Main Theorem.

By the compact form of the Gruenberg–Kegel graph $\Gamma(G)$ for a group G we mean a graph whose vertices are labeled with sets of primes. A vertex labeled by a set τ represents the clique of $\Gamma(G)$ such that every vertex in this clique is labeled by a prime from τ. An edge connecting two sets represents the set of edges of $\Gamma(G)$ that connect each vertex in the first set with each vertex in the second one.

Lemma 3.1 ([35, Proposition 2.7] and [34, Proposition 3.2]) *Let $G \cong F_4(q)$, where $q = 2^n$, $r, s \in \pi(G)$ and $r \neq s$. Then r and s are nonadjacent if and only if one of the following conditions holds* (*up to permutation*)*:*

(1) $2 = r$, *and* $e(s, q) \in \{8, 12\}$.
(2) $s, r \neq 2$, $k = e(r, q)$, $l = e(s, q)$, $1 \leqslant k < l$ *and either* $l \in \{8, 12\}$, *or* $l = 6$ *and* $k \in \{3, 4\}$, *or* $l = 4$ *and* $k = 3$.

In particular, the compact form for $\Gamma(F_4(q))$ *is the following:*

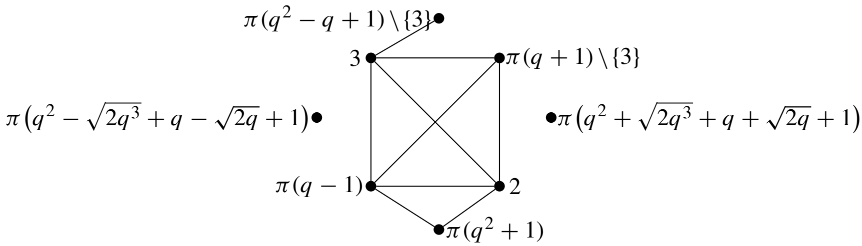


Remark 3.2 Note that $R_6(2)$ and $R_1(2)$ are empty sets.

Lemma 3.3 ([34, Proposition 3.3] and [35, Proposition 2.9]; see also [9, Lemma 3]) *Let* $G \cong {}^2F_4(q)$*, where* $q = 2^{2n+1}$*,* $r, s \in \pi(G)$ *and* $r \neq s$*. Put* $m_1(n) = q - 1$*,* $m_2(n) = q + 1$*,* $m_3(n) = q^2 + 1$*,* $m_4(n) = q^2 - q + 1$*,* $m_5(n) = q^2 - \sqrt{2q^3} + q - \sqrt{2q} + 1$*,* $m_6(n) = q^2 + \sqrt{2q^3} + q + \sqrt{2q} + 1$*. Then* r *and* s *are nonadjacent in* $\Gamma(G)$ *if an only if one of the following conditions holds:*

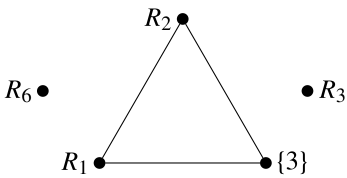
(1) $2 = r$*,* s *divides* $m_k(n)$*,* $s \neq 3$*, and* $k > 3$*.*
(2) $2 \neq s, r$*; either* $3 \neq r \in \pi(m_k(n))$*,* $3 \neq s \in \pi(m_l(n))$ *for* $k \neq l$*, and* $\{k, l\} \neq \{1, 2\}, \{1, 3\}$*; or* $r = 3$ *and* $s \in \pi(m_l(n))$*, where* $l \in \{3, 5, 6\}$*.*

In particular, the compact form for $\Gamma({}^2F_4(q))$ *is the following:*



Lemma 3.4 ([35, Proposition 2.7] and [34, Propositions 3.2, 4.5]) *Let* $G \cong G_2(q)$*, where* $q = 3^k$*,* $r, s \in \pi(G)$ *and* $r \neq s$*. Then* r *and* s *are nonadjacent in* $\Gamma(G)$ *if and only if* $e(r, q)$ *or* $e(s, q)$ *belongs to* $\{3, 6\}$*.*

In particular, the compact form for $\Gamma(G_2(q))$ *is the following:*

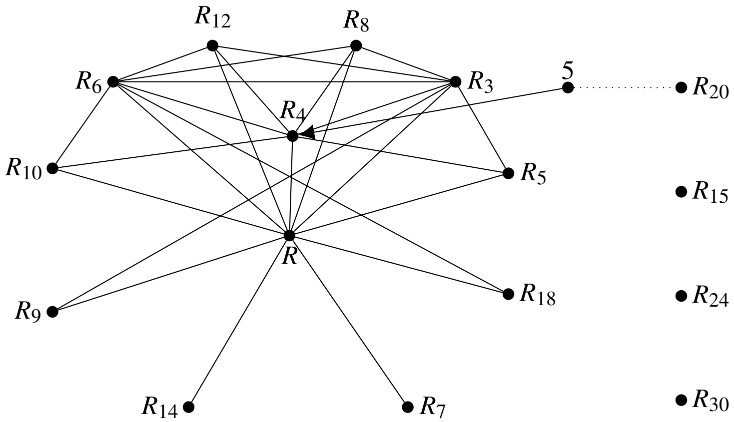


Remark 3.5 Note that the set $R_1(3)$ is empty.

Lemma 3.6 ([35, Proposition 2.7] and [34, Propositions 3.2, 4.5]) *Let $G \cong E_8(q)$, where q is a power of a prime p. Suppose that $r, s \in \pi(G)$ with $r \neq s$. Then r and s are nonadjacent in $\Gamma(G)$ if and only if one of the following conditions holds:*

(1) $r \in \{2, p\}$, $s \neq p$ *and* $e(s, q) \in \{15, 20, 24, 30\}$.
(2) $s, r \notin \{2, p\}$, $k = e(r, q)$, $l = e(s, q)$, $1 \leqslant k < l$, *and either* $l = 6$ *and* $k = 5$, *or* $l \in \{7, 14\}$ *and* $k \geqslant 3$, *or* $l = 9$ *and* $k \geqslant 4$, *or* $l \in \{8, 12\}$ *and* $k \geqslant 5$, $k \neq 6$, *or* $l = 10$ *and* $k \geqslant 3$, $k \notin \{4, 6\}$, *or* $l = 18$ *and* $k \notin \{1, 2, 6\}$, *or* $l = 20$ *and* $r \cdot k \neq 20$, *or* $l \in \{15, 24, 30\}$.

In particular, the compact form for $\Gamma(E_8(q))$ is the following. Here, $R(q) = R_1(q) \cup R_2(q) \cup \{p\}$ and the vector from 5 to $R_4(q)$ and the dotted edge $\{5, R_{20}(q)\}$ indicate that $R_4(q)$ and $R_{20}(q)$ are not adjacent, but if $5 \in R_4(q)$ (i.e., $q^2 \equiv -1 \pmod 5$), then there exist edges between 5 and the primes from $R_{20}(q)$.



4 Almost recognizability of groups $G_2(q)$, $F_4(q)$, and $^2F_4(q)$ by Gruenberg–Kegel graph

In this section, we prove Main Theorem for groups $L = F_4(q)$, where $q \geqslant 2$ is a power of 2, $L = {}^2F_4(q)$, where $q = 2^{2m+1} > 2$, and $L = G_2(q)$, where $q > 3$ is a power of 3.

Theorem 4.1 *If G is a group with $\Gamma(G) = \Gamma(L)$, then $L \cong \mathrm{Inn}(L) \trianglelefteq G \leqslant \mathrm{Aut}(L)$. In particular, L is almost recognizable by Gruenberg–Kegel graph.*

By Lemmas 3.1, 3.3, and 3.4 , we see that $t(L) \geqslant 3$ and $t(2, L) \geqslant 2$. It follows from Lemma 2.8 that there exists a nonabelian simple group S such that $S \cong \mathrm{Inn}(S) \leqslant G/K \leqslant \mathrm{Aut}(S)$, where K is the solvable radical of G. Moreover, by the Thompson theorem on finite groups with fixed-point-free automorphisms of prime order [30, Theorem 1], K is nilpotent. To prove Theorem 4.1, it suffices to show that $S \cong L$ and $K = 1$. These two facts are established in the following four lemmas.

Lemma 4.2 $S \cong L$.

Proof If $q > 2$ is even and $L = {}^2F_4(q)$ or $L = F_4(q)$, then the lemma follows from [1, Theorem 3.4] and [20, Theorem 3.3], respectively.

Suppose that $L = F_4(2)$. Then $\pi(S) \subseteq \pi(L) = \{2, 3, 5, 7, 13, 17\}$. Note that $13 \in R_{12}(2)$ and $17 \in R_8(2)$. By Lemma 3.1, we find that 13 and 17 are nonadjacent to all vertices in $\Gamma(G)$. Therefore, $\{13, 17\} \subset \pi(S)$ by Lemma 2.8. Inspecting [38, Table 1], we see that $S \in \{\mathrm{PSU}_4(4), \mathrm{PSU}_3(17), \mathrm{PSL}_2(13^2), \mathrm{PSp}_4(13), \mathrm{PSL}_3(16), \mathrm{PSp}_6(4), P\Omega_8^+(4), F_4(2)\}$. Using [5, Corollary 3], we find that 5 and 17 are adjacent in $\Gamma(\mathrm{PSL}_2(13^2))$ and $\Gamma(\mathrm{PSL}_3(16))$, while 3 and 17 are adjacent in $\Gamma(\mathrm{PSU}_3(17))$ and $\Gamma(\mathrm{PSU}_4(4))$. According to [6, Corollaries 2–4], 5 and 17 are adjacent in $\Gamma(\mathrm{PSp}_4(13))$, $\Gamma(\mathrm{PSp}_6(4))$, and $\Gamma(P\Omega_8^+(4))$. This implies that $S \cong F_4(2)$, as claimed.

It remains to consider the case $L \cong G_2(q)$, where $q > 3$ is a power of 3. By [40, Theorem 1.1], we can assume that $q = 3^{2k}$ for a positive integer k. By Lemma 3.4, sets $R_3(q)$ and $R_6(q)$ are connected components in $\Gamma(L)$. It follows from Lemma 2.8 that sets $R_3(q)$ and $R_6(q)$ are connected components in $\Gamma(S)$ and hence $\Gamma(S)$ has at least three connected components. Therefore, S is isomorphic to a group listed in Introduction before Main Theorem. We show that $S \cong L$ considering each case for S separately. We will extensively use that $k_3(q) = \frac{q^2+q+1}{(q-1,3)} = q^2 + q + 1$ and $k_6(q) = \frac{q^2-q+1}{(q+1,3)} = q^2 - q + 1$ (see equation (1)).

Case $S \cong A_p$, where $p > 6$ and both p and $p - 2$ are primes. The connected components of $\Gamma(S)$ are $\pi_1(S)$, $\{p\}$, and $\{p - 2\}$. We know that $R_3(q)$ and $R_6(q)$ are connected components in $\Gamma(S)$, so either $p - 2 \in R_3(q)$ or $p - 2 \in R_6(q)$. It follows from Lemma 2.3 that $p - 3$ is divisible by 3; a contradiction since $p \neq 3$.

Case $S \cong E_8(u)$, where u is a prime power. Since $3^{2k} \equiv (-1)^k \pmod 5$, we infer that $q \equiv \pm 1 \pmod 5$. This implies that there exists $\varepsilon \in \{+, -\}$ such that $q^2 + \varepsilon q + 1 \equiv 3 \pmod 5$. Take any prime divisor r of $k_3(\varepsilon q) = q^2 + \varepsilon q + 1$ such that $r \not\equiv 1 \pmod 5$. Since $r \in R_3(\varepsilon q)$, Lemma 3.6 implies that $r \in R_j(u)$, where $j \in \{15, 20, 24, 30\}$. Then $r - 1$ is divisible by j and hence $j = 24$. Since $R_{24}(u)$ is a connected component in $\Gamma(S)$, we find that $R_3(\varepsilon q) = R_{24}(u)$. If $K \neq 1$, then Lemma 2.16 implies that $R_{24}(u) \subseteq \pi_1(G)$; a contradiction since $r \notin \pi_1(G)$. Assume that there exists $s \in \pi(\overline{G}/S)$ such that $s > 3$. It follows from Lemma 2.8 that $s \in \pi_1(G)$. By [12, Theorem 2.5.12, Definition 2.5.13], there exists an element $g \in G \setminus S$ such that $|g| = s$ and g acts on S as a field automorphism. It is known that $C_S(g) \cong E_8(u^{1/s})$ (see, e.g., [12, Proposition 4.9.1]). By Lemma 2.2, we see that $R_{24}(u^{1/s}) \subseteq R_{24}(u)$ and hence s is adjacent to r in $\Gamma(G)$; we arrive at a contradiction since $s \in \pi_1(G)$ and $r \notin \pi_1(G)$. Now consider a coclique $\rho = \{s_1, s_2, \ldots, s_{12}\}$ of maximal size in $\Gamma(S)$. By Lemma 3.6, we have $(s_i, 6) = 1$ for every $i \in \{1, \ldots, 12\}$. Applying Lemma 2.7, we find that $t(G) \geqslant 12$; a contradiction with Lemma 3.4.

Cases $S \in \{\mathrm{PSU}_6(2), \mathrm{PSL}_3(4), M_{11}, M_{23}, M_{24}, J_3, \mathrm{HiS}, \mathrm{Suz}, \mathrm{Co}_2, \mathrm{Fi}_{23}, F_2, M_{22}, \mathrm{Fi}_{24}', F_1\}$. By Lemma 2.8, we infer that $\{2, r_3(q), r_6(q)\}$ is a coclique in $\Gamma(S)$. It follows from Lemma 2.3 that $r_3(q) \equiv 1 \pmod 6$ and $r_6(q) \equiv 1 \pmod 6$. Using [8], we see that there is no such a coclique in $\Gamma(S)$; a contradiction.

Case $S \cong F_4(u)$, where u is even. Since $q^2+q+1 \equiv 3 \pmod 8$, there exists $r \in R_3(q)$ such that $r \not\equiv 1 \pmod 8$. By Lemmas 2.8, 3.1, and 2.3 , we infer that $r \in R_{12}(u)$. Then $\pi(q^2 + q + 1) = R_{12}(u)$. On the other hand, we have $q^2 + q + 1 \equiv 7 \pmod{12}$

and hence there exists $s \in \pi(q^2+q+1)$ such that $s \not\equiv 1 \pmod{12}$; a contradiction with Lemma 2.3.

Case $S \cong {}^2F_4(u)$, *where* $u = 2^{2m+1} > 2$. By Lemma 3.3, it is true that $\pi_2(S) \cup \pi_3(S) = \pi(u^4-u^2+1) \subseteq \pi(u^6+1)$. Consider any prime $r \in R_3(q)$. Then $r \in \pi_2(S) \cup \pi_3(S)$ and r divides $(u^3)^2+1$. Since $(u^3)^2 \equiv -1 \pmod r$, we find that -1 is a quadratic residue modulo r. This implies that $r \equiv 1 \pmod 4$. Since r is an arbitrary element of $R_3(q)$, we infer that $q^2+q+1 \equiv 1 \pmod 4$. On the other hand, $q^2+q+1 \equiv 3^{4k}+3^{2k}+1 \equiv 3 \pmod 4$; a contradiction.

Case $S \cong {}^2B_2(u)$, *where* $u = 2^{2m+1} > 2$. According to [2, Table 3], we can assume that $\pi_2(S) = \pi(u-1)$, $\pi_3(S) = \pi(u-\sqrt{2u}+1)$, and $\pi_4(S) = \pi(u+\sqrt{2u}+1)$. Note that $\pi(u-\sqrt{2u}+1) \cup \pi(u+\sqrt{2u}+1) = \pi(u^2+1)$. If $R_3(q) \neq \pi_2(S)$, then $R_3(q) \subseteq \pi(u^2+1)$ and we get a contradiction arguing as in the case $S \cong {}^2F_4(u)$. Therefore, we can assume that $R_3(q) = \pi(u-1)$. Suppose that $r \mid u-1 = 2^{2m+1}-1$. Then $r \mid 2^{2n+2}-2$ and hence 2 is a quadratic residue modulo r. This implies that $r \equiv \pm 1 \pmod 8$. It follows that $q^2+q+1 \equiv \pm 1 \pmod 8$; a contradiction since $q^2+q+1 \equiv 3^{4k}+3^{2k}+1 \equiv 3 \pmod 8$.

Case $S \cong {}^2G_2(u)$, *where* $u = 3^{2m+1} > 3$. According to [2, Table 2], we can assume that $\pi_2(S) = \pi(u-\sqrt{3u}+1)$ and $\pi_3(S) = \pi(u+\sqrt{3u}+1)$. Take any prime $r \in R_{12k}(3)$. By Lemma 2.2, we infer that $r \in R_6(q)$. We know that $R_6(q) \subseteq \pi_2(S) \cup \pi_3(S) = \pi(u^2-u+1) = R_6(u)$. By Lemma 2.4, we conclude that $12k \mid 6 \cdot (2m+1)$; a contradiction.

Case $S \cong {}^2D_p(3)$, *where* $p = 2^m+1 \geqslant 3$ *is prime*. According to [2, Table 2], we can assume that $\pi_2(S) = \pi((3^{p-1}+1)/2)$ and $\pi_3(S) = \pi((3^p+1)/4)$. Take any $r \in R_{12k}(3)$. By Lemma 2.2, we infer that $r \in R_6(q)$. Since $r \in \pi_2(S) \cup \pi_3(S)$, we find that r divides $3^{2(p-1)}-1$ or $3^{2p}-1$. It follows from Lemma 2.4 that $12k \mid 2(p-1)$ or $12k \mid 2p$; a contradiction.

Case $S \cong A_1(u) \cong \mathrm{PSL}_2(q)$, *where* $u = 2^m > 2$. According to [2, Table 2], we can assume that $\pi_2(S) = \pi(u-1)$ and $\pi_3(S) = \pi(u+1)$. Note that 3 divides u^2-1. On the other hand, $\pi(u^2-1) = \pi_2(S) \cup \pi_3(S) = \pi_2(G) \cup \pi_3(G)$; a contradiction since $3 \in \pi_1(G)$.

Case $S \cong A_1(u) \cong \mathrm{PSL}_2(u)$, *where* $3 < u = v^n$ *and* u *is odd*. Consider $\varepsilon \in \{+,-\}$ such that $u \equiv \varepsilon 1 \pmod 4$. According to [2, Table 2], we can assume that $\pi_1(S) = \pi(u-\varepsilon 1)$, $\pi_2(S) = \{v\}$, and $\pi_3(S) = \pi\left(\frac{u+\varepsilon 1}{2}\right)$. Therefore, there exists $\tau \in \{+,-\}$ such that $\pi(q^2+\tau q+1) = \{v\}$ and $\pi(q^2-\tau q+1) = \pi\left(\frac{u+\varepsilon 1}{2}\right)$. Moreover, we know that $\pi(u-\varepsilon 1) \subseteq \pi(3(q^2-1))$. Since $q^2-q+1 = (q-1)^2+(q-1)+1$, Lemma 2.5 implies that either $q^2+\tau q+1 = v$ or $q = 19$ and $v = 7$. By assumption, $q = 3^{2k}$ and hence $q^2+\tau q+1 = v$. Then $v-1$ is divisible by 9. Therefore, $v \geqslant 19$ and $3 \in \pi\left(\frac{u-1}{2}\right) \setminus \pi(q^2-\tau q+1)$. This implies that $\varepsilon = +$ since, by Lemma 3.4, $3 \in \pi_1(G)$.

Suppose that n is even. Then v^2-1 divides $u-1$, so $\pi(v+1) \subseteq \pi(q^2-1)$. Take any $r \in \pi(v+1)$. Since $q \equiv \pm 1 \pmod r$ and r divides $q^2+\tau q+2$, we find that r divides 3 ± 1 and hence $r = 2$. Therefore, $q^2+\tau q+2$ is a power of 2. On the other hand, $q^2+\tau q+2 \equiv 1 \pm 1 + 2 \pmod 8$. This implies that $q^2+\tau q+2 \leqslant 4$; a contradiction.

We can assume that n is odd. Then $\frac{v+1}{2}$ divides $\frac{u+1}{2}$. Take any $r \in \pi\left(\frac{v+1}{2}\right)$. Then r divides both $q^2 + \tau q + 2$ and $q^2 - \tau q + 1$. Therefore, $2\tau q \equiv -1 \pmod r$ and hence $0 \equiv 4q^2 + 4\tau q + 8 \equiv 4\tau q + 9 \equiv 7 \pmod r$. This implies that $r = 7$ and $v+1 = 2\cdot 7^m$ for a positive integer m. Since $q \equiv 1 \pmod 8$ and $q^2+\tau q+2 = 2\cdot 7^m$, we infer that $3 + \tau 1 \equiv 2\cdot(-1)^m \pmod 8$. This implies that $\tau = -1$ and m is even. A straightforward calculation shows that $9^{2k} - 9^k + 2$ is not divisible by 49 for all positive integer k; a contradiction.

Cases $S \in \{F_3, \mathrm{O'N}, J_1, \mathrm{LyS}, J_4, {}^2E_6(2), E_7(2), E_7(3)\}$. According to [2, Tables 2, 3], we see that there exist primes r and s such that $\pi(q^2+q+1) = \{r\}$ and $\pi(q^2-q+1) = \{s\}$. Applying Lemma 2.6, we find that $r \equiv s \equiv 1 \pmod 6$ and the remainders of r and s divided by 8 belong to the set $\{1, 3\}$. Inspecting [2, Tables 2, 3], we see that in each case there are no two primes satisfying these restrictions; a contradiction.

Thus, we conclude that $S \cong G_2(u)$, where u is a power of 3. Now Lemma 2.1 implies immediately that $u = q$ and, therefore, $S \cong L$. This completes the proof of the lemma. □

Remark 4.3 For $q = 3$, the paper [40] contains a mistake, and $S \in \{G_2(3), \mathrm{PSL}_2(13)\}$ by [19, Table 1].

Show that $K = 1$. Consider a minimal (by order) counterexample G to this claim.

Lemma 4.4 *K is an elementary abelian r-group for some $r \in \pi_1(L)$.*

Proof Let r be a prime divisor of $|K|$. Since K is nilpotent, we have a decomposition $K = P \times U$, where P is a Sylow r-subgroup of K and U is a maximal normal subgroup of K such that $r \notin \pi(U)$. Since U and $\Phi(P)$ are characteristic subgroups of K, the subgroup $N = U \times \Phi(P)$ is a normal subgroup of G. Then $\Gamma(G/N)$ is a subgraph of $\Gamma(G)$. On the other hand, $\Gamma(L)$ is a subgraph of $\Gamma(G/N)$ and hence $\Gamma(G/N) = \Gamma(L)$. By the minimality of G, we infer that $N = 1$ and, therefore, K is an elementary abelian r-group. □

By Lemmas 2.9, 4.2 and 4.4 , we can assume that $G = K \rtimes X$, where K is an elementary abelian r-group for $r \in \pi_1(L)$ and $\mathrm{Soc}(X) = S \cong L$.

Lemma 4.5 $K = 1$ *if* $L = F_4(q)$.

Proof According to [24, Table 5.1], S has a subgroup H isomorphic to ${}^3D_4(q)$.

Assume that $r \notin \{2\} \cup R_{12}(q)$. By Lemma 2.12, r is adjacent to each element from $R_{12}(q)$ in $\Gamma(KH)$. It follows from Lemma 3.1 that $\Gamma(G) \neq \Gamma(L)$; a contradiction.

Assume that $r = 2$. By Lemma 2.11, S is unisingular and, therefore, 2 is adjacent to all other vertices in $\Gamma(KS)$. We arrive at a contradiction with Lemma 3.1.

Assume that $r \in R_{12}(q)$. Since q is even, by [24, Table 5.1], S has a subgroup $H_1 \cong P\Omega_8^+(q)$. Now by [4, Table 8.50], H_1 has a subgroup

$$H_2 \cong P\Omega_4^-(q) \times P\Omega_4^-(q) \cong \mathrm{PSL}_2(q^2) \times \mathrm{PSL}_2(q^2).$$

Therefore, for each $s \in R_4(q)$, a Sylow s-subgroup of S is non-cyclic. This implies that r and s are adjacent in $\Gamma(KH_2)$ (see, for example, [11, Theorem 10.3.1]). Therefore, r and s are adjacent in $\Gamma(G)$; a contradiction with Lemma 3.1. Thus, $K = 1$. □

Lemma 4.6 *$K = 1$ if $L \cong G_2(q)$ for $q > 3$ or $L \cong {}^2F_4(q)$ for $q > 2$.*

Proof Let $p = 2$ if $L \cong {}^2F_4(q)$ and $p = 3$ if $L \cong G_2(q)$. By Lemma 2.11, L is unisingular and, therefore, if $r = p$, then $\Gamma(G)$ is connected; a contradiction. Therefore, $r \in \pi_1(L) \setminus \{p\}$. By Lemmas 2.13 and 2.14 , r is adjacent to a prime from $\pi_2(L)$; a contradiction. Thus, $K = 1$. $\square$

This completes the proof of Theorem 4.1.

Since the group L is known to be almost recognizable, the following natural question arises.

Problem 4.7 Suppose that L is a group from the statement of Theorem 4.1. Find a positive integer k such that L is k-recognizable by Gruenberg–Kegel graph.

5 Unrecognizability of groups ${}^2B_2(q)$ and $G_2(3)$ by Gruenberg–Kegel graphs

In this short section, we show that groups ${}^2B_2(q)$ with $q > 2$ and $G_2(3)$ are unrecognizable.

Proposition 5.1 *Let $G = {}^2B_2(q)$, where $q > 2$ is an odd power of 2. Then G is unrecognizable by Gruenberg–Kegel graph.*

Proof By [15, Lemma 3.6], there exists a 4-dimensional module V over the field of order q such that each nontrivial element from each maximal torus of G acts fixed-point freely on V. Thus, $\Gamma(V \rtimes G) = \Gamma(G)$ and, therefore, G is unrecognizable by [7, Theorem 1.2]. $\square$

Proposition 5.2 *Let $G \cong G_2(3)$. Then G is unrecognizable by Gruenberg–Kegel graph.*

Proof Using [8], we find that $\Gamma(G_2(3))$ is the following:

$$2 \bullet\!\!\longrightarrow\!\!\bullet\, 3 \qquad \bullet\, 7 \qquad \bullet\, 13$$

Moreover, $\Gamma(G) = \Gamma(\mathrm{PSL}_2(13))$. By [17, p. 9] and Lemma 2.15, there is a 6-dimensional irreducible $\mathrm{PSL}_2(13)$-module V over a field of characteristic two such that all elements in $\mathrm{PSL}_2(13)$ of orders 7 and 13 act fixed-point freely on V. Therefore,

$$\Gamma(V \rtimes \mathrm{PSL}_2(13)) = \Gamma(\mathrm{PSL}_2(13)) = \Gamma(G).$$

Thus, by [7, Theorem 1.2], G is unrecognizable by Gruenberg–Kegel graph. $\square$

6 Almost recognizability of groups $E_8(q)$ by Gruenberg–Kegel graph

To complete the proof of Main Theorem, it remains to consider the case of groups $E_8(q)$. In [39], Zavarnitsine proved that if G is a finite group such that $\Gamma(G) =$

$\Gamma(E_8(q))$, where $q \equiv 0, \pm 1 \pmod 5$, then $G \cong E_8(u)$ for some $u \equiv 0, \pm 1 \pmod 5$. The aim of this section is to prove the following similar result for the remaining cases of q.

Theorem 6.1 *Let $L = E_8(q)$, where $q \equiv \pm 2 \pmod 5$ is a prime power, and G be a group such that $\Gamma(G) = \Gamma(L)$. Then $G \cong E_8(u)$ for some prime power u with $u \equiv \pm 2 \pmod 5$.*

Proof Since $\Gamma(G)$ is disconnected, Lemma 2.8 implies that there exists a nonabelian simple group S such that $S \leqslant G/K \leqslant \mathrm{Aut}(S)$, where K is the solvable radical of G. By the Thompson theorem on finite groups with fixed-point-free automorphisms of prime order [30, Theorem 1], K is nilpotent. By Lemmas 2.8 and 3.6, we find that $s(S) \geqslant s(L) = 4$, $t(2, S) \geqslant t(2, L) = 5$ and $t(S) \geqslant t(L) - 1 = 11$. According to [2, Table 1] and [34, Tables 2, 4, and 5], we find that either $S \cong F_1$ or $S \cong E_8(u)$, where u is a prime power.

Suppose that $S \cong F_1$. According to [2, Table 3], we see that $s(S) = 4$, for each $i \geqslant 2$, $|\pi_i(S)| = 1$, and $\pi(S) \setminus \pi_1(S) = \{41, 59, 71\}$. At the same time, $\pi_2(G) = R_{15}(q)$, $\pi_3(G) = R_{24}(q)$, and $\pi_4(G) = R_{30}(q)$. By Lemma 2.3, the numbers 15, 24, and 30 divide $r_i - 1$ for pairwise distinct primes r_i from $\pi(S) \setminus \pi_1(S)$; a contradiction.

Suppose that $S \cong E_8(u)$, where u is a power of a prime v and $u \equiv 0, \pm 1 \pmod 5$. Since $q \equiv \pm 2 \pmod 5$, we find that $5 \in R_4(q)$. Denote

$$\theta = \big\{r_9(q), r_{14}(q), r_7(q), r_{18}(q), r_{15}(q), r_{24}(q), r_{30}(q)\big\}.$$

By Lemma 3.6, we see that $\theta \cup \{5\}$ is a coclique of size 8 in $\Gamma(G)$. It follows from Lemma 2.8 that at least six elements of θ belong to $\pi(S)$. Take any $r \in \pi(S) \cap \theta$. Since r and 5 are nonadjacent in $\Gamma(G)$, they are nonadjacent in $\Gamma(S)$. We know that $5 \in \{v\} \cup R_1(u) \cup R_2(u)$ and hence $r \in R_{20}(u) \cup R_{15}(u) \cup R_{24}(u) \cup R_{30}(u)$ according to Lemma 3.6. This implies that at least two elements from $\theta \cap \pi(S)$ are adjacent in $\Gamma(S)$; a contradiction.

Suppose that $S \cong E_8(u)$, where $v^l = u \equiv \pm 2 \pmod 5$. According to [2, Table 3], we can assume that $\pi_2(G) = R_{15}(q)$, $\pi_3(G) = R_{24}(q)$, and $\pi_4(G) = R_{30}(q)$, while $\pi_2(S) = R_{15}(u)$, $\pi_2(S) = R_{24}(u)$, and $\pi_4(S) = R_{30}(u)$. If $K \neq 1$, then Lemma 2.16 implies that $R_{24}(u) \subset \pi_1(G)$; a contradiction since $R_{24}(u)$ must coincide with a connected component of $\Gamma(G)$ not containing 2. Therefore, we can assume that $S \leqslant G \leqslant \mathrm{Aut}(S)$.

Now prove that $G/S = 1$. If a prime r divides $|G : S|$, then by [12, Theorem 2.5.12, Definition 2.5.13, Proposition 4.9.1], $G \setminus S$ contains a field automorphism x of S of order r with $C_S(x) \geqslant E_8(u^{1/r})$. This implies that r is adjacent in $\Gamma(G)$ to each prime from $\pi(E_8(u^{1/r}))$. Suppose that $r \notin \{2, 3, 5\}$. By Lemma 2.2, $\Gamma(G)$ is connected and hence $\Gamma(G) \neq \Gamma(S)$. If $r = 2$, then by Lemma 2.2, $R_{15}(u) \subset \pi_1(G)$; a contradiction. If $r = 5$, then by Lemma 2.2, r is adjacent in $\Gamma(G)$ to some primes from $R_{24}(u)$ and hence $R_{24}(u) \subset \pi_1(G)$; a contradiction. Therefore, we can assume that G/S is a 3-group. By Lemma 3.6, we see that $r_i(q)$ is nonadjacent to 2 in $\Gamma(G)$ if and only if $i \in \{15, 20, 24, 30\}$. Similarly, a prime $r_i(u)$ is nonadjacent to 2 in $\Gamma(S)$ if and only if $i \in \{15, 20, 24, 30\}$. Since $R_{20}(q) \subset \pi_1(G)$ and $R_{20}(u) \subset \pi_1(G)$, we infer that

$R_{20}(q) = R_{20}(u)$. By Lemma 2.2, we infer that 3 is adjacent in $\Gamma(G)$ to a prime from $R_{20}(u)$, therefore, $\Gamma(G) \neq \Gamma(S)$.

Thus, $G = S$. The proof of Theorem 6.1 is complete. □

Corollary 6.2 *For each value of q, the group $E_8(q)$ is almost recognizable by Gruenberg–Kegel graph.*

Proof If G is a group such that $\Gamma(G) = \Gamma(E_8(q))$, then by [39, Theorem 1] and Theorem 6.1, we have $G \cong E_8(u)$ for a prime power u. By Lemma 2.10, for a given q, the number of possibilities for u is finite. This implies that there is only the finite number of possibilities for G (up to isomorphism), in particular, $E_8(q)$ is almost recognizable by Gruenberg–Kegel graph. □

Problem 6.3 Do there exist prime powers q and q_1 with $q \neq q_1$ and $\Gamma(E_8(q)) = \Gamma(E_8(q_1))$?

Corollary 6.4 *If G is a finite group such that $\Gamma(G) = \Gamma(E_8(q))$ and $|G| = |E_8(q)|$ for some prime power q, then $G \cong E_8(q)$.*

Proof By Theorem 6.1, if $\Gamma(G) = \Gamma(E_8(q))$, then $G \cong E_8(q_1)$ for some prime power q_1. It is clear that the function

$$f(x) = x^{120}(x^2-1)(x^8-1)(x^{12}-1) \cdot (x^{14}-1)(x^{18}-1)(x^{20}-1)(x^{24}-1)(x^{30}-1)$$

strictly monotonically increases if $x \geqslant 1$. Thus, if

$$f(q_1) = |G| = |E_8(q)| = f(q),$$

then $q_1 = q$, and, therefore, $G \cong E_8(q)$. □

In [33], it was proved that if G is a simple group and H is a group such that $\omega(H) = \omega(G)$ and $|H| = |G|$, then $H \cong G$. Thus, each simple group is uniquely determined by its order and spectrum. It is known that if q is odd and $n \geqslant 3$, then $\Gamma(P\Omega_{2n+1}(q)) = \Gamma(\mathrm{PSp}_{2n}(q))$ and $|P\Omega_{2n+1}(q)| = |\mathrm{PSp}_{2n}(q)|$ but these groups are not isomorphic. Therefore, it is natural to consider the following problem.

Problem 6.5 For which simple groups G is the following true: if H is a group with $\Gamma(H) = \Gamma(G)$ and $|H| = |G|$, then H is isomorphic to G?

Problem 6.5 was formulated by Khosravi in his survey paper [18, Question 4.2], by Kondrat'ev in frame of the open problems session of the 13th School-Conference on Group Theory Dedicated to Vyacheslav A. Belonogov's 85th Birthday (see [25, Question 4]), and was independently formulated by Wujie Shi in a personal communication with the first author. Also Problem 6.5 was formulated in the paper by Cameron and the first author (see [7, Problem 2]). It is clear that if a simple group is quasirecognizable by Gruenberg–Kegel graph, then Problem 6.5 solves in the positive for this group. At the same time, Corollary 6.4 gives a solution of Problem 6.5 for finite simple groups $E_8(q)$ which are not necessary quasirecognizable by Gruenberg–Kegel graph.

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Declarations

Conflict of interest The authors declare no competing interests.

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