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Coloured-noise-induced transport in a model of the thermochemical reactor

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In this paper, effects of coloured noise on the stochastic excitement in a model of the thermochemical flow reactor are studied. Transport phenomena associated with noise-induced generation of large-amplitude oscillations are investigated depending on the correlation time of coloured noise. We study how probability of the noise-induced excitement is related to the stochastic sensitivity of the system to coloured noise with certain correlation characteristics. Parameter zones of the high stochastic sensitivity are found and discussed in connection with occurrence of resonance.

This article is part of the theme issue 'Transport phenomena in complex systems (part 2)'.

1. Introduction

Transport phenomena in complex nonlinear dynamic systems attract a considerable amount of interest from researchers in the different domains of science and engineering. The inevitably present random disturbances, being an additional factor complicating the transport processes, generates new, often counterintuitive noise-induced phenomena [1–8]. Nowadays, in the investigations of these phenomena, along with direct numerical simulations [9], a new approach based on the stochastic sensitivity analysis is actively elaborated [10–16].

In studies of stochastic effects, white Gaussian noises are commonly used as a model of random disturbances. But white noise, which is an idealization of fluctuations, is not always realistic and can lead to inaccurate conclusions [17]. The lack of temporal correlation in white noise is too strong an assumption in many cases. Therefore, it is important to model noise with certain correlation time. Coloured noise

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is one of the widely used models where the characteristic time defines the decay of the correlation function, and the correlation time is a key parameter.

Specific responses of systems to coloured noises have been found and studied in different branches of natural science, for example, in biochemistry [18], lasers [19,20], volcanic dynamics [21], seismic activity [22], population dynamics [23,24], microbiology [25] and cancer dynamics [26].

Coloured noise can generate a wide variety of stochastic phenomena: stochastic and coherence resonance [27,28], synchronization [29,30], phase transitions [31,32], noise-induced bifurcations [33] and order-chaos transformations [34]. In the present paper, we study the effects of coloured noises in the thermochemical reactions. As a basic deterministic model, we consider a system suggested in [35] to describe thermokinetics of a homogeneous dilute mixture of gases in the surrounding thermostat. In this model, stochastic transport in the form of the generation of mixed-mode oscillations by white Gaussian noise was studied in [15]. The present paper is devoted to the study of new features of stochastic excitation associated with the specifics of more complex coloured noises operating in the system.

In §2, for the considered model, we give a short description of deterministic dynamics in parameter zones of mono- and bistability with attractors in the form of the equilibria and limit cycles. Detailed presentation and analysis of coloured-noise-induced excitement is given in §3. Here, the dependence of stochastic effects on the correlation time of coloured noise is demonstrated numerically and studied analytically on the basis of stochastic sensitivity technique [36].

2. Deterministic dynamics

Consider a model of the thermochemical reactor with ideal mixing [35,37] in the form of the system of two differential equations

$$\left.\begin{aligned} \dot{x} &= \sqrt{y}\left(-x\exp\left(-\frac{\delta}{y}\right) + p\,(1-x)\right) \\ \text{and} \qquad \dot{y} &= \frac{2}{3}q\sqrt{y}\left(x\exp\left(-\frac{\delta}{y}\right) + r\,(1-y)\right), \end{aligned}\right\} \tag{2.1}$$

where the dimensionless variables x and y describe dynamics of the concentration of the reactant and the temperature, correspondingly.

In this paper, we fix $p = 0.24$, $r = 0.0622$, $\delta = 5$ and change the parameter q. Under these assumptions, the model (2.1) is quite representative and exhibits various key regimes of two-dimensional system dynamics. For any q, the system (2.1) has the equilibrium $M(\bar{x}, \bar{y})$ with $\bar{x} = 0.875207$, $\bar{y} = 1.481516$.

In the variation of dynamical behaviour of the system (2.1), points $q_1 = 55.363$ and $q_2 = 68.244$ play a key role. Point q_1 marks the saddle-node bifurcation, and q_2 corresponds to the subcritical Hopf bifurcation (see bifurcation diagram in figure 1). For $q < q_2$, the equilibrium M is stable, and unstable otherwise. At the point q_1, a limit cycle is born that is stable for $q > q_1$. In the present paper, we focus on the zone $50 \le q \le q_2$. For $50 < q < q_1$, the system (2.1) is monostable with the stable equilibrium M as a single attractor. In the interval $q_1 < q < q_2$, the system (2.1) is bistable and possesses two attractors: the stable equilibrium M and the stable limit cycle Γ.

In figure 2, we present two typical examples of phase trajectories for these two parameter zones.

Figure 2*a* for $q = 55$ shows characteristic features of the phase trajectories for the monostability zone. Here, for small deviations from the stable equilibrium (filled circle), the trajectory immediately tends to the equilibrium. If the deviation exceeds a certain threshold, the trajectory first goes away from the equilibrium, makes large-amplitude loops, and only after that begins to approach M. In figure 2*b* for $q = 68$ we illustrate phase trajectories for the bistability zone where the stable equilibrium M is shown by a filled circle, and the stable cycle Γ is plotted in green.

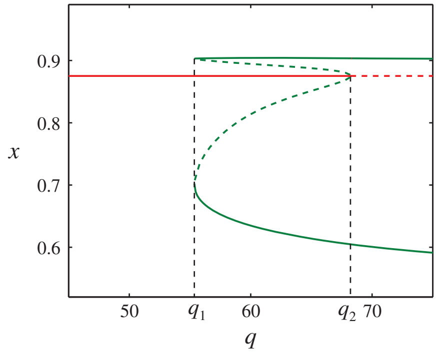

Figure 1. Attractors and repellers of the deterministic system (2.1). Stable (unstable) equilibria are shown by solid (dashed) red lines; stable (unstable) limit cycles are shown by solid (dashed) green lines. (Online version in colour.)

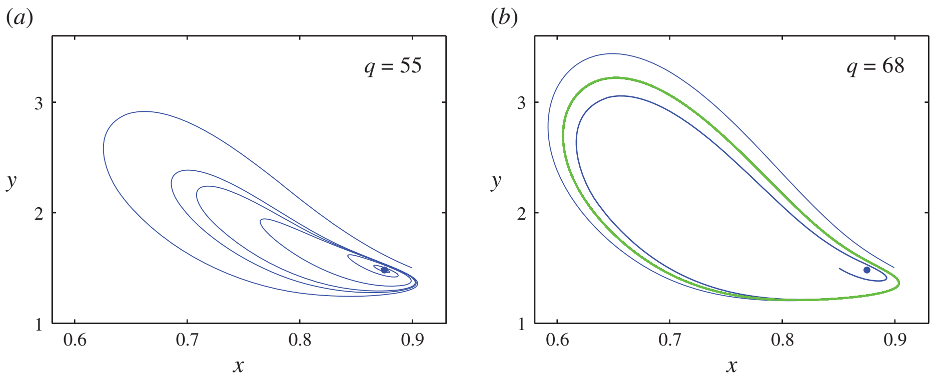

Figure 2. Phase portraits of the deterministic system (2.1): (*a*) for $q = 55$, (*b*) $q = 68$. Here, the stable equilibrium M is shown by a filled circle, and the stable limit cycle Γ is plotted in green. (Online version in colour.)

Here, for small deviations from M, the trajectory immediately tends to the equilibrium whereas for larger deviations the trajectory tends to the orbit of the stable cycle Γ.

So, for both mono- and bistability cases, there are sub- and superthreshold zones in the phase plane. Such behaviour is typical for excitable systems. In the presence of random disturbances, such systems exhibit a special type of stochastic transport, namely noise-induced excitement with sharp transitions from small-amplitude oscillations around the stable equilibrium to large-amplitude stochastic loops. This phenomenon for the stochastic variant of system (2.1) with random disturbances modelled by white Gaussian noise was studied in [15]. In the present paper, we will study important additional features of the stochastic excitement for random disturbances modelled by coloured noise.

3. Stochastic dynamics

Consider the system (2.1) with additional random disturbances:

$$\left.\begin{aligned}\dot{x} &= \sqrt{y}\left(-x\exp\left(-\frac{\delta}{y}\right) + p\,(1-x)\right)\\ \text{and}\qquad \dot{y} &= \frac{2}{3}q\sqrt{y}\left(x\exp\left(-\frac{\delta}{y}\right) + r\,(1-y)\right) + \varepsilon s(t).\end{aligned}\right\} \tag{3.1}$$

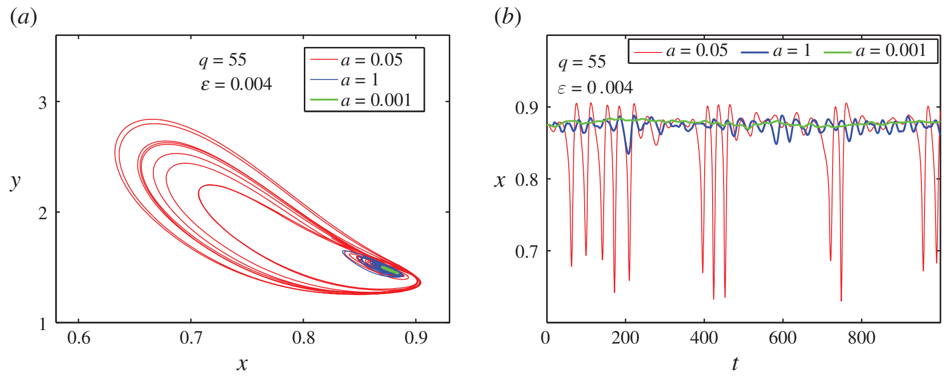

Figure 3. Stochastic system (3.1), (3.2) with $q = 55$ and $\varepsilon = 0.004$: (*a*) phase trajectories and (*b*) time series. (Online version in colour.)

Here, the stochastic forcing $\eta(t) = \varepsilon s(t)$ of intensity ε is modelled by coloured noise $s(t)$ with parameters

$$\langle s(t)\rangle = 0, \quad \langle s(t)s(t')\rangle = \exp(-a|t - t'|),$$

where the parameter $a = 1/\tau$ is defined by correlation time τ.

This coloured noise $s(t)$ can be simulated by the following Langevin equation:

$$\dot{s} = -as + \sqrt{2a}\xi(t). \tag{3.2}$$

Here, $\xi(t)$ is the standard uncorrelated Gaussian white noise with parameters $\langle\xi(t)\rangle = 0$, $\langle\xi(t)\xi(t')\rangle = \delta(t - t')$. So, the stochastic term $\eta(t)$ in (3.1) has the variation $D(\eta) = \langle\eta^2(t)\rangle = \varepsilon^2$.

Consider how the phenomenon of the stochastic transport in the form of noise-induced excitement in system (3.1) depends on the parameters q, a and ε. In numerical simulations of stochastic solutions of the system (3.1), we used the Euler–Maruyama scheme with the time step 10^{-4}.

In figure 3, we show a behaviour of the stochastic system (3.1) with the parameter $q = 55$ lying in the monostability zone where the equilibrium M is a single attractor. Here, phase trajectories (figure 3*a*) and time series (figure 3*b*) of solutions starting at the equilibrium M are plotted for fixed noise intensity $\varepsilon = 0.004$ and different values of the parameter a of the correlation time.

As can be seen, random trajectories for $a = 1$ (blue) and $a = 0.001$ (green) exhibit small-amplitude stochastic fluctuations near the equilibrium M. For $a = 0.05$, trajectories (red) demonstrate an excitable stochastic behaviour with the alternation of large-amplitude spikes and small-amplitude fluctuations near M. So, the phenomenon of stochastic excitement significantly depends on the correlation parameter a of coloured noise (3.2).

In figure 4, we plot phase trajectories (figure 4*a*) and time series (figure 4*b*) of stochastic systems (3.1), (3.2) with the parameter $q = 68$ from the bistability zone where the equilibrium M coexists with the stable limit cycle Γ. Here, solutions starting at the equilibrium M are plotted for smaller noise intensity $\varepsilon = 0.001$ and different values of the correlation parameter a.

In this case, random trajectories for $a = 10$ (blue) and $a = 0.001$ (green) slightly fluctuate near the equilibrium M. For $a = 0.1$, trajectories (red) transit to the basin of the cycle Γ and continue to oscillate near Γ. As can be seen, in this case, the phenomenon of noise-induced excitement is also observed and depends on the correlation parameter a. Note that for $q = 68$, this phenomenon occurs under smaller noise than for $q = 55$.

Details of stochastic transport of random states under increasing noise intensity are shown in figure 5*a* for $q = 55$ and in figure 5*b* for $q = 68$. A sharp increase in variance and a shift in mean values are clearly seen here.

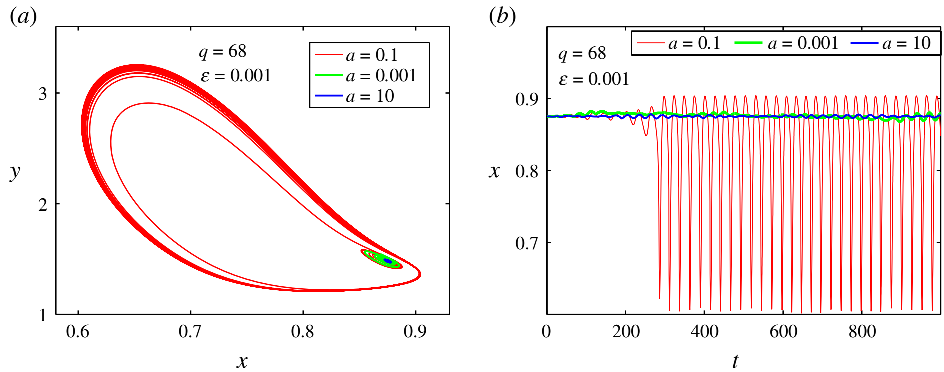

Figure 4. Stochastic system (3.1), (3.2) with $q = 68$ and $\varepsilon = 0.001$: (*a*) phase trajectories and (*b*) time series. (Online version in colour.)

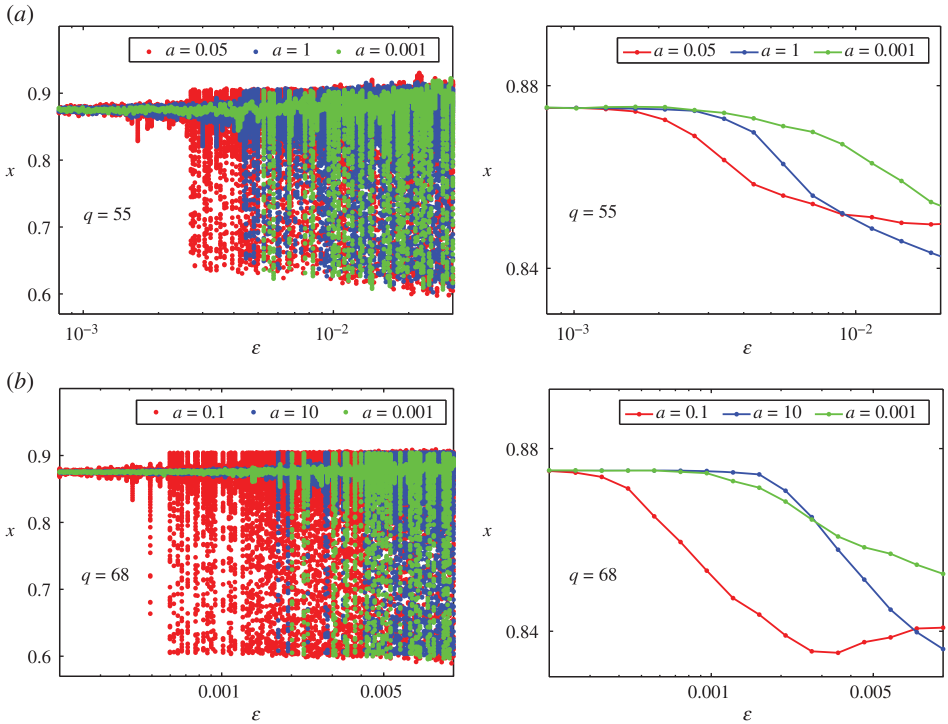

Figure 5. Random states (left) and mean values (right) of the stochastic system (3.1),(3.2) for different values of the parameter *a* versus noise intensity: (*a*) for $q = 55$ and (*b*) for $q = 68$. (Online version in colour.)

For the parametric statistical analysis of the noise-induced excitement, we studied the probability $\mathcal{P}$ of the exit of trajectories starting at M to the zone $x \leq \tilde{x}$ during the time interval $0 \leq t \leq T$. In our analysis, we put the threshold value $\tilde{x} = 0.75$, and $T = 200$.

Plots of the function $\mathcal{P}(a)$ are shown in figure 6 for various values of q and ε. As can be seen, plots $\mathcal{P}(a)$ are not monotonous and have peaks. These peaks are localized in a certain a-parameter zone. It is the coloured noise with such a correlation parameter a that excites the system. So, the response of the stochastic system (3.1) to the coloured noise significantly depends on the parameter a.

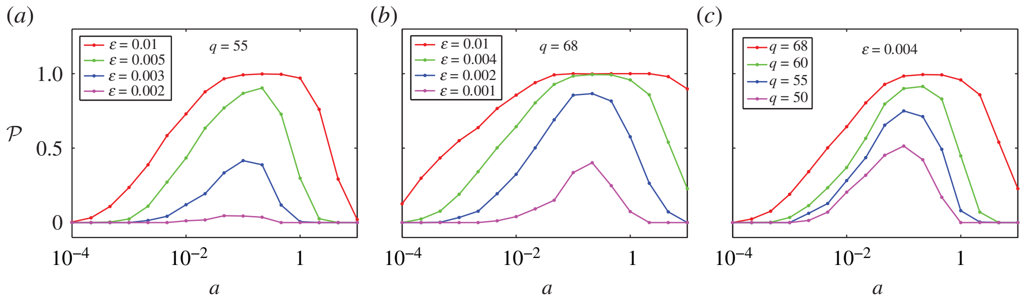

Figure 6. Probability of noise-induced excitement: (*a*) for $q = 55$, (*b*) for $q = 68$, (*c*) for $\varepsilon = 0.004$ and various q. (Online version in colour.)

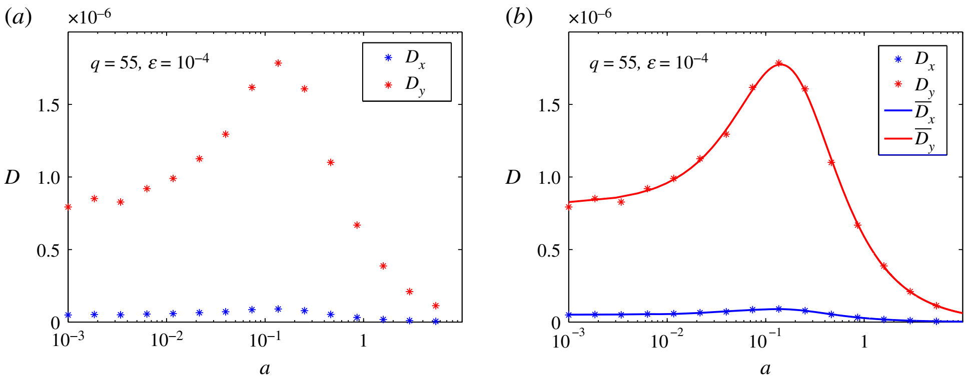

Figure 7. Stochastic system with $q = 55$: (*a*) variances D_x and D_y, (*b*) comparison of variances with approximations $\bar{D}_x$ and $\bar{D}_y$. (Online version in colour.)

To study the dependence of this response on the time correlation parameter a, consider variances $D_x = \langle (x - \bar{x})^2 \rangle$ and $D_y = \langle (y - \bar{y})^2 \rangle$ which describe dispersions of coordinates near the equilibrium $M(\bar{x}, \bar{y})$. In figure 7*a*, values of the variances D_x (blue) and D_y (red) found by direct numerical simulation of random solutions of the system (3.1) with $q = 55$, $\varepsilon = 10^{-4}$ are shown by asterisks. As one can see, peaks of the functions $\mathcal{P}(a)$ and $D_{x,y}(a)$ are observed for the same a-parameter zone where the system is most susceptible to noise.

To study this susceptibility to coloured noise analytically, we will use the stochastic sensitivity function technique. A key asymptotic characteristic of the random states deviations from the stable equilibrium is the stochastic sensitivity matrix W. This matrix is a solution [36,38] of the algebraic equation (A 6) (see appendix), where F is the Jacobi matrix at the equilibrium, $\sigma = 1$, the matrix G characterizes random disturbances, and a is the correlation parameter of coloured noise. For system (3.1), (3.2), we have

$$W = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}, \quad F = \begin{bmatrix} \alpha & \beta \\ q\gamma & q\mu \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad GG^\top = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

where

$$\alpha = \sqrt{y}\left(-\exp\left(-\frac{\delta}{y}\right) - p\right),$$

$$\beta = \frac{1}{2\sqrt{y}}\left(-x\exp\left(-\frac{\delta}{y}\right) + p\,(1-x)\right) - \frac{\delta x}{y\sqrt{y}}\exp\left(-\frac{\delta}{y}\right)$$

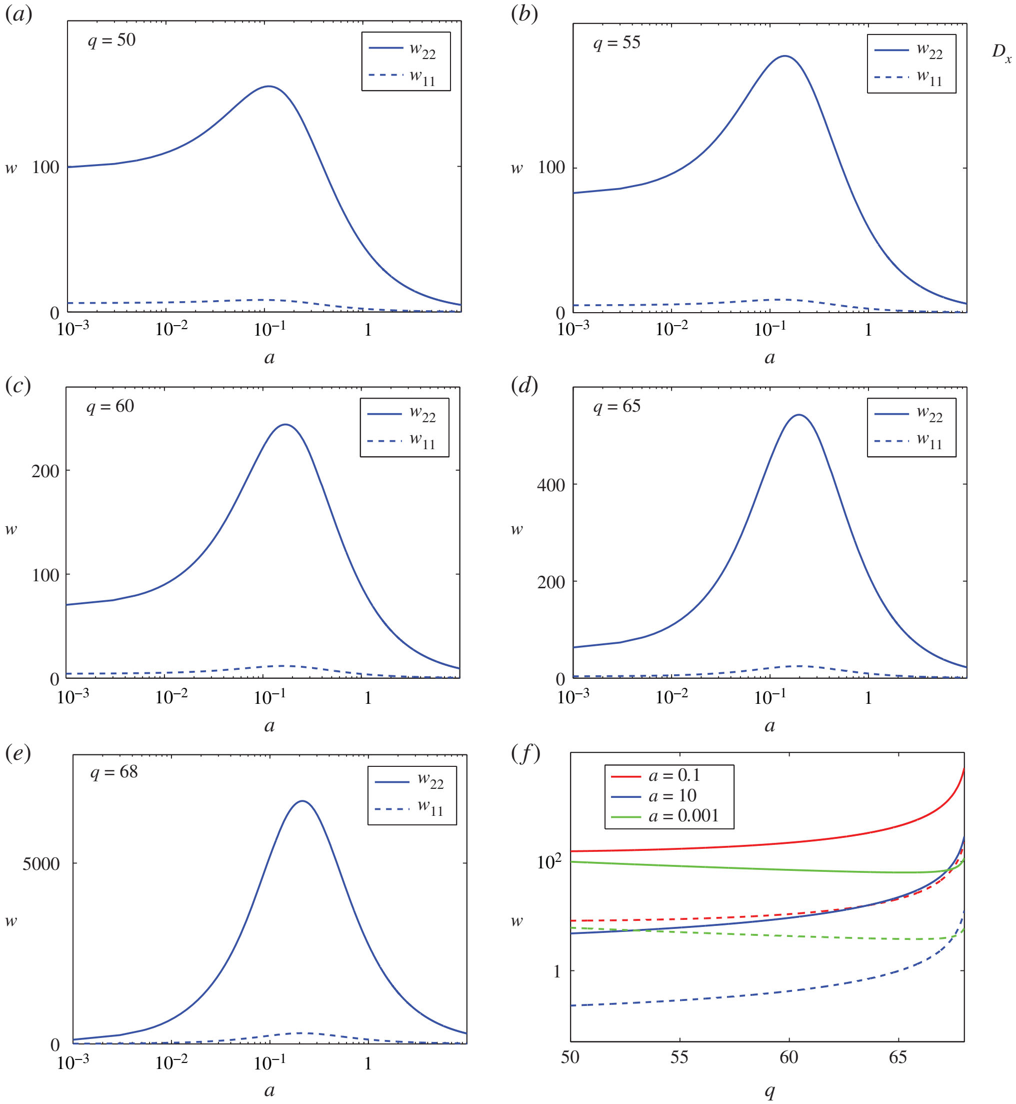

Figure 8. Stochastic sensitivity of the equilibrium versus parameter a for (a) $q = 50$, (b) $q = 55$, (c) $q = 60$, (d) $q = 65$, (e) $q = 68$, and versus q in (f) for $a = 0.1$ (red), $a = 10$ (blue), $a = 0.001$ (green).

$$\gamma = \frac{2\sqrt{y}}{3}\exp\left(-\frac{\delta}{y}\right),$$

$$\mu = \frac{1}{3\sqrt{y}}\left(x\exp\left(-\frac{\delta}{y}\right) + r(1-y)\right) + \frac{2\sqrt{y}}{3}\left(\frac{\delta x}{y^2}\exp\left(-\frac{\delta}{y}\right) - r\right).$$

For elements of the stochastic sensitivity matrix W, we take into account $w_{12} = w_{21}$ and rewrite the matrix equation (A6) as a system of three scalar equations:

$$\left.\begin{aligned} &\alpha w_{11} + \beta w_{12} = 0 \\ &q\gamma w_{11} + (\alpha + q\mu)w_{12} + \beta w_{22} = -\frac{\beta}{\Delta} \\ \text{and}\qquad &q\gamma w_{12} + q\mu w_{22} = \frac{\alpha - a}{\Delta}, \end{aligned}\right\} \tag{3.3}$$

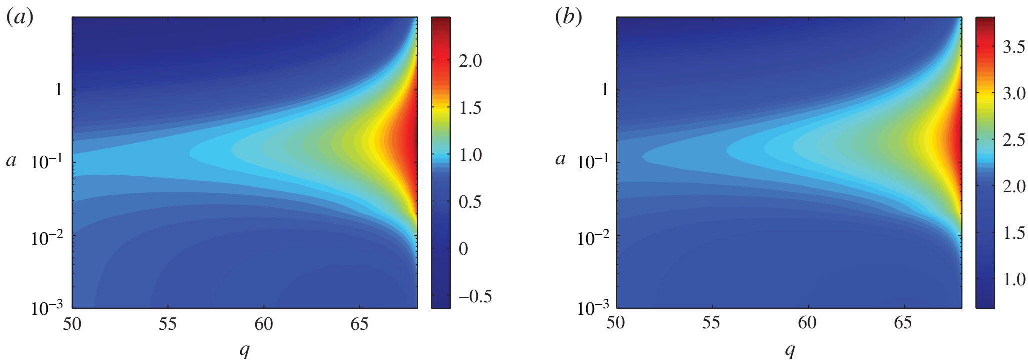

Figure 9. Stochastic sensitivity of the equilibrium M in system (3.1): (*a*) $\log_{10} w_{11}(q,a)$, (*b*) $\log_{10} w_{22}(q,a)$.

where $\Delta = a^2 - (\alpha + q\mu)a + q(\alpha\mu - \beta\gamma)$. Note that the equilibrium M of the deterministic system (2.1) is exponentially stable if and only if $\mathrm{tr}F = \alpha + q\mu < 0$ and $\det F = q(\alpha\mu - \beta\gamma) > 0$, so $\Delta > 0$.

The system (3.3) has the solution

$$w_{11} = \frac{\beta^2(q\mu + \alpha - a)}{q\Delta(\alpha + q\mu)(\alpha\mu - \beta\gamma)},$$

$$w_{12} = -\frac{\alpha}{\beta} w_{11}$$

and

$$w_{22} = \frac{\alpha\beta q\gamma + (\alpha - a)(\alpha^2 + \alpha q\mu - \beta q\gamma)}{q\Delta(\alpha + q\mu)(\alpha\mu - \beta\gamma)}.$$

For parameters $p = 0.24$, $r = 0.0622$, $\delta = 5$, considered in this paper, we have

$$\alpha = -0.3337748, \quad \beta = -0.08304464, \quad \gamma = 0.02776849, \quad \mu = 0.00489089.$$

Elements of the stochastic sensitivity matrix W give the following approximations:

$$D_x = \langle (x - \bar{x})^2 \rangle \approx \varepsilon^2 w_{11}, \quad \langle (x - \bar{x})(y - \bar{y}) \rangle \approx \varepsilon^2 w_{12}, \quad D_y = \langle (y - \bar{y})^2 \rangle \approx \varepsilon^2 w_{22}.$$

As can be seen in figure 7*b* for $q = 55$, analytical approximations $\overline{D}_x(a) = \varepsilon^2 w_{11}(a)$ and $\overline{D}_y(a) = \varepsilon^2 w_{22}(a)$ agree well with values of variances D_x and D_y.

The dependence of the stochastic sensitivity of the equilibrium on the parameters a and q is illustrated in figure 8. Here, dashed lines show w_{11} and solid lines show w_{22}. In figure 8*a–e*, functions $w_{11}(a)$ and $w_{22}(a)$ are plotted for different values of q. As can be seen, values $w_{22}(a)$ are significantly larger than $w_{11}(a)$. These functions essentially depend on a. The distinctive feature of these plots is the existence of sharp peaks near $a = 0.1$. This means that the stochastic sensitivity of the system to coloured noise is defined by the correlation time. As can be seen in figure 8*f*, stochastic sensitivity increases unlimitedly as parameter q approaches the bifurcation point q_2.

More details can be seen in the two-parameter (q, a)-diagram in figure 9 where functions $\log_{10} w_{11}(q,a)$ and $\log_{10} w_{22}(q,a)$ are shown by colour. These figures allow us to assume that the system (3.1) is highly sensitive to coloured noise with the correlation parameter a close to 0.1. As a consequence of such a high sensitivity, the system exhibits the phenomenon of stochastic excitement presented above in figures 3–6.

Values of the parameter a corresponding to the peaks of the stochastic sensitivity and, hence, to the phenomenon of noise-induced excitement, can be interpreted as resonance values.

4. Conclusion

It is known that in nonlinear dynamical systems coloured noise can result in specific transport phenomena. In the present paper, we studied how the variation of correlation time of coloured noise influences on the phenomenon of stochastic excitement in a model of the thermochemical flow reactor. Probabilistic analysis has shown that stochastic excitability depends significantly on the correlation characteristics of the acting noise. For the theoretical investigation of this dependence, we have developed a method that uses the apparatus of stochastic sensitivity. For the time correlation parameter, a narrow zone of the high stochastic sensitivity was found. It was shown that it is in this parameter zone that resonance phenomena of the noise-induced excitement appear.

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Authors' contributions. All authors contributed equally to the present research article.
Competing interests. The authors declare that they have no competing interests.
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Appendix A

Consider a general n-dimensional nonlinear dynamical system

$$\dot{x} = f(x, r) \tag{A 1}$$

with the m-vector parameter r of random disturbances. Let the equilibrium $\bar{x}$ be exponentially stable in the unforced system with $r = 0$. It is assumed that random disturbances $r(t) = \varepsilon s(t)$, $s = (s_1, \dots, s_m)^\top$ are modelled by the following stochastic equations:

$$\dot{s}_i = -a_i s_i + \sigma_i \sqrt{2a_i}\xi_i(t), \quad a_i > 0 \tag{A 2}$$

forced by standard uncorrelated Gaussian white noises $\xi_i(t)$ $(i = 1, \dots m)$ with parameters $\langle \xi_i(t) \rangle = 0$, $\langle \xi_i(t)\xi_i(t') \rangle = \delta(t - t')$. Here, $r(t)$ is a coloured noise of the noise intensity ε, and $s_i(t)$ are scalar uncorrelated coloured noises with parameters

$$\langle s_i(t) \rangle = 0, \quad \langle s_i(t)s_i(t') \rangle = \sigma_i^2 \exp(-a_i|t - t'|), \; \tau_i = \frac{1}{a_i}.$$

Parameters a_i define correlation times τ_i of these coloured noises.

For asymptotics $y(t) = \lim_{\varepsilon \to 0}((x^\varepsilon(t) - \bar{x})/\varepsilon)$ of deviations of the solution $x^\varepsilon(t)$ of the stochastic dynamical system

$$\dot{x} = f(x, \varepsilon s)$$

from the equilibrium $\bar{x}$, one can write the stochastic linear extension system

$$\dot{y} = Fy + Gs$$

and

$$\dot{s} = -As + C\xi(t).$$

Here,

$$F = \frac{\partial f}{\partial x}(\bar{x}, 0), \quad G = \frac{\partial f}{\partial r}(\bar{x}, 0)$$

and

$$A = \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_m \end{bmatrix}, \quad C = \begin{bmatrix} \sigma_1\sqrt{2a_1} & & 0 \\ & \ddots & \\ 0 & & \sigma_m\sqrt{2a_m} \end{bmatrix}, \; \xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_m \end{bmatrix}.$$

For the matrix $Z = \mathrm{E}\, zz^\top$ of second moments for the extended $(n+m)$-dimensional vector $z = \left[\begin{smallmatrix} y \\ s \end{smallmatrix}\right]$, one can write the system [36,38]

$$\dot{Z} = \Phi Z + Z\Phi^\top + SS^\top, \quad \Phi = \begin{bmatrix} F & G \\ O & -A \end{bmatrix}, \; S = \begin{bmatrix} O \\ C \end{bmatrix}. \tag{A 3}$$

For the stable $\bar{x}$ and positive a_i, the system (A3) has a unique stable stationary solution Z satisfying the following matrix equation:

$$\Phi Z + Z\Phi^\top + SS^\top = 0.$$

For blocks $W = \mathrm{E}\, yy^\top$, $M = \mathrm{E}\, ys^\top$, $B = \mathrm{E}\, ss^\top$ of the matrix $Z = \begin{bmatrix} W & M \\ M^\top & B \end{bmatrix}$, one can write the following system:

$$\left.\begin{aligned} &FW + WF^\top + GM^\top + MG^\top = 0 \\ &FM + GB - MA = 0 \\ \text{and}\quad &AB + BA = 2AQ. \end{aligned}\right\} \tag{A 4}$$

Here, $Q = \mathrm{diag}[\sigma_1^2, \ldots, \sigma_m^2]$. Excluding $B = Q$ from the third equation of (A4), we have

$$\left.\begin{aligned} &FW + WF^\top + GM^\top + MG^\top = 0 \\ \text{and}\quad &FM + GQ - MA = 0. \end{aligned}\right\} \tag{A 5}$$

The matrix W defines the stochastic sensitivity of the equilibrium $\bar{x}$, and can be used for approximation

$$\mathrm{cov}(x^\varepsilon(t), x^\varepsilon(t)) \approx \varepsilon^2 W.$$

In the case when the coloured noise s is scalar and modelled by the single equation

$$\dot{s} = -as + \sigma\sqrt{2a}\,\xi(t),$$

we have $Q = \sigma^2$, $A = a$ and $M = -\sigma^2 (F - aI)^{-1} G$. In this case, the stochastic sensitivity matrix W can be found from the equation

$$FW + WF^\top = \sigma^2 [GG^\top (F^\top - aI)^{-1} + (F - aI)^{-1} GG^\top]. \tag{A 6}$$

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