

Cross-connection structure of locally inverse semigroups

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Locally inverse semigroups are regular semigroups whose idempotents form pseudo-semilattices. We characterize the categories that correspond to locally inverse semigroups in the realm of Nambooripad's cross-connection theory. Further, we specialize our cross-connection description of locally inverse semigroups to inverse semigroups and completely 0-simple semigroups, obtaining structure theorems for these classes. In particular, we show that the structure theorem for inverse semigroups can be obtained using only one category, quite analogous to the Ehresmann–Schein–Nambooripad Theorem; for completely 0-simple semigroups, we show that cross-connections coincide with structure matrices, thus recovering the Rees Theorem by categorical tools.

Keywords: Locally inverse semigroup; inverse semigroup; completely 0-simple semigroup; category; cross-connection; Ehresmann–Schein–Nambooripad theorem; Rees theorem.

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1. Introduction

Nambooripad [23], building on an approach initiated by Hall [14] and Grillet [12], introduced the notion of a normal category as the abstract categorical model of principal one-sided ideals of a regular semigroup. Each regular semigroup S gives rise to two normal categories: one that models the principal left ideals of S and another one that corresponds to the principal right ideals. Then Nambooripad devised a structure called *cross-connection* that captured the nontrivial interrelation between these two normal categories. Cross-connections form a category, and Nambooripad proved that this category is equivalent to the category of regular semigroups.

Thus, regular semigroups have precise categorical counterparts. A natural research program is to explore this link between semigroups and categories, by browsing through interesting families of regular semigroups and looking how their properties manifest in the language of cross-connections. Within this framework, Nambooripad's school studied cross-connections of transformation semigroups of certain concrete species, see, e.g. [5, 6, 8, 31]. However, we believe that, taking into account the abstract character of the category-based approach, it is more promising to look at specializations of Nambooripad's correspondence to important **abstract** classes of regular semigroups. To the best of our knowledge, results in this direction restrict to the following so far: the first-named author and Rajan [7] characterized cross-connections of completely simple semigroups and Rajan *et al.* [30] studied cross-connections in normal bands. Another closely related work was also by Rajan [29] wherein some properties of the normal categories of the inverse semigroups were studied, though a complete characterization was not achieved.

Here, we characterize cross-connections of *locally inverse semigroups*. There are several reasons for us to focus on this particular class. First of all, locally inverse semigroups were introduced and deeply studied by Nambooripad himself [21, 22] (under the name *pseudo-inverse semigroups*) so that re-visiting them appears to be appropriate for a paper dedicated to his memory. Second, the class of locally inverse semigroups is of interest and importance from both the structural viewpoint and the viewpoint of the theory of so-called e-varieties of regular semigroups. We refer to Pastijn [26] and McAlister [20] for elegant covering theorems that reveal the structure of locally inverse semigroups. As for the e-varietal viewpoint, the e-variety of all locally inverse semigroups is distinguished in many aspects; in particular, it is one of the two largest e-varieties which admit bi-free objects, see Yeh [33]. Last but not least, we have anticipated that the categorical counterpart of such a well-behaved sort of regular semigroups should be nice in a sense, and we believe that the present paper provides some supporting evidence for these expectations.

In Sec. 2, we characterize the categories of principal one-sided ideals of locally inverse semigroups as so-called unambiguous categories and describe a category equivalence between the category of cross-connected unambiguous categories and the category of locally inverse semigroups. In Sec. 3, we specialize our construction to inverse semigroups to provide a new structure theorem for inverse semigroups and

also retrieve (a weaker version of) the Ehresmann–Schein–Nambooripad Theorem. This can be seen as completing the line of research initiated in [29]. In Sec. 4, we relate the cross-connection structure of completely 0-simple semigroups to the Rees Theorem; it turns out that cross-connections are nothing but sandwich-matrices. This generalizes the results of [7] and also gives a concrete illustration for the abstract discussion of Sec. 2. We conclude in Sec. 5 by outlining directions for future work.

We assume some familiarity with basic notions from category theory (such as functors and natural transformations) and semigroup theory (such as Green’s relations). For undefined notions, we refer to [15, 18] for category theory and [11, 13, 16] for semigroups. Inevitably, we have followed the general line of and used several results from Nambooripad’s treatise [23] but we have made a fair effort to make the present paper understandable without studying [23] in detail.

2. Cross-Connections of Locally Inverse Semigroups

2.1. Inverse and locally inverse semigroups

Two elements a, b of a semigroup S are said to be *inverses* of each other if $aba = a$ and $bab = b$. A semigroup is called *regular* if each of its elements has an inverse. *Inverse semigroups* are defined as semigroups in which every element has a unique inverse; given an element x in such a semigroup, its unique inverse is denoted x^{-1} .

An element e of a semigroup S is called an *idempotent* if $e = e^2$, and we let $E(S)$ stand for the set of all idempotents of S . A *band* is a semigroup in which all elements are idempotents and a *semilattice* is a commutative band.

Recall a classic characterization of inverse semigroups (see, e.g. [13, Theorem II.2.6]).

Theorem 2.1. *Let S be a semigroup. The following are equivalent:*

- (1) S is an inverse semigroup.
- (2) Every \mathcal{R} -class of S contains exactly one idempotent and every \mathcal{L} -class of S contains exactly one idempotent.
- (3) S is regular and $E(S)$ is a semilattice.

A regular semigroup S is *locally inverse* if eSe is an inverse semigroup for each $e \in E(S)$. We need further characterizations of locally inverse semigroups; for this purpose, we require some more definitions.

Given a semigroup S , we define two preorders \leq_ℓ and \leq_r on the set $E(S)$ as follows: $e \leq_\ell f$ if and only if $ef = e$ and $e \leq_r f$ if and only if $fe = e$. The equivalences \mathcal{L} and \mathcal{R} on $E(S)$ induced by the preorders \leq_ℓ and \leq_r , respectively, are nothing but the restrictions of Green’s classic equivalences on S . The intersection $\leq := \leq_\ell \cap \leq_r$ is a partial order on $E(S)$. We denote by $\omega(e)$ the order ideal of

the set $(E(S), \leq)$ generated by e , that is, $\omega(e) := \{f \in E(S) \mid f \leq e\}$. Similarly, $\omega^\ell(e) := \{f \in E(S) \mid f \leq_\ell e\}$ and $\omega^r(e) := \{f \in E(S) \mid f \leq_r e\}$. A band is said to be *left [right] normal* if it satisfies $xyz = xzy$ [respectively, $xyz = yxz$].

The following fact comes from [21; 26, Result 1].

Theorem 2.2. *Let S be a regular semigroup. The following are equivalent:*

- (1) S is a locally inverse semigroup;
- (2) if $e, f, g \in E(S)$ are such that $e (\mathcal{L} \cup \mathcal{R}) f$ and $e, f \in \omega(g)$, then $e = f$;
- (3) for each $e \in E(S)$, the set $\omega(e)$ is a semilattice;
- (4) for each $e \in E(S)$, the set $\omega^\ell(e)$ forms a left normal band and the set $\omega^r(e)$ forms a right normal band.

2.2. Normal categories

For a small category \mathcal{C} , its set of objects is denoted by $v\mathcal{C}$ while its set of morphisms is denoted by just \mathcal{C} . For $c, d \in v\mathcal{C}$, the set of all morphisms from c to d is denoted by $\mathcal{C}(c, d)$. We compose functions and morphisms from left to right so that in expressions like $\varphi\psi$ or $\gamma * \delta$ etc., the left factor applies first. The identity morphism at an object $c \in v\mathcal{C}$ is denoted 1_c .

A morphism $\varphi : c \rightarrow d$ is called an *epimorphism* if it is left-cancellative, i.e. for all morphisms $\alpha, \beta : d \rightarrow e$, the equality $\varphi\alpha = \varphi\beta$ implies $\alpha = \beta$. Similarly, $\varphi : c \rightarrow d$ is called a *monomorphism* if it is right-cancellative, i.e. for all morphisms $\alpha, \beta : b \rightarrow c$, the equality $\alpha\varphi = \beta\varphi$ implies $\alpha = \beta$. A morphism $\varphi : c \rightarrow d$ is called an *isomorphism* if $\varphi\psi = 1_c$ and $\psi\varphi = 1_d$ for some morphism $\psi : d \rightarrow c$.

A *preorder* is a category with at most one morphism from an object to another. A *strict preorder* is a preorder whose only isomorphisms are the identity morphisms.

A *subcategory* of a category \mathcal{C} is a category \mathcal{D} with $v\mathcal{D} \subseteq v\mathcal{C}$ and such that $\mathcal{D}(a, b) \subseteq \mathcal{C}(a, b)$ for all $a, b \in v\mathcal{D}$, with the same identities and composition of morphisms. \mathcal{D} is a *full subcategory* of \mathcal{C} if $\mathcal{D}(a, b) = \mathcal{C}(a, b)$ for all $a, b \in v\mathcal{D}$.

Definition 2.1. Let \mathcal{C} be a small category and \mathcal{P} a subcategory of \mathcal{C} with $v\mathcal{P} = v\mathcal{C}$. The pair $(\mathcal{C}, \mathcal{P})$ (often denoted by just \mathcal{C}) is a *category with subobjects* if:

- (1) \mathcal{P} is a strict preorder.
- (2) Every morphism in \mathcal{P} is a monomorphism in \mathcal{C} .
- (3) If $\varphi, \psi \in \mathcal{P}$ and $\varphi = \alpha\psi$ for some $\alpha \in \mathcal{C}$, then $\alpha \in \mathcal{P}$.

In a category $(\mathcal{C}, \mathcal{P})$ with subobjects, the morphisms in \mathcal{P} are called *inclusions*. If $c' \rightarrow c$ is an inclusion, we write $c' \subseteq c$ and we denote this inclusion by $\iota(c', c)$. An inclusion $\iota(c', c)$ *splits* if there exists $\theta \in \mathcal{C}(c, c')$ such that $\iota(c', c)\theta = 1_{c'}$, and then the morphism θ is called a *retraction*.

Definition 2.2. Let \mathcal{C} be a category with subobjects. A *normal factorization* of a morphism $\varphi \in \mathcal{C}$ is a decomposition of the form $\varphi = \theta\sigma\iota$, where θ is a retraction, σ is an isomorphism and ι is an inclusion in \mathcal{C} . The morphism $\theta\sigma$ is called the

epimorphic component of the morphism φ and is denoted by φ° . The codomain of φ° is called the *image* of φ and the codomain of the retraction θ is called the *coimage* of φ . We denote the coimage of φ by $\text{coim } \varphi$.

Remark 2.3. In general, a normal factorization of a morphism (even when it exists) is not unique. It may be noted that the coimage depends on the normal factorization chosen and therefore may vary. However, as shown in [23, Proposition II.5], for a morphism with a normal factorization, it has a unique epimorphic component and hence a unique image. In particular, the epimorphic component (and hence the image) is independent of the chosen normal factorization.

We record the following result which will be needed in the sequel.

Proposition 2.4 ([23, Proposition II.7]). *Let \mathcal{C} be a category with subobjects such that every morphism has a normal factorization. Then:*

- (1) *the inclusion of any epimorphism $\varphi \in \mathcal{C}$ is the identity morphism and every normal factorization of φ is of the form $\varphi = \theta\sigma$, where θ is a retraction and σ is an isomorphism;*
- (2) *the retraction of any monomorphism $\psi \in \mathcal{C}$ is the identity morphism and every normal factorization of ψ is of the form $\psi = \sigma\iota$, where σ is an isomorphism and ι is an inclusion.*

Definition 2.3. Let \mathcal{C} be a category with subobjects and $d \in v\mathcal{C}$. A *cone with apex d* is a function $\gamma : v\mathcal{C} \rightarrow \mathcal{C}$ such that:

- (1) for each $a \in v\mathcal{C}$, one has $\gamma(a) \in \mathcal{C}(a, d)$;
- (2) $\iota(a, b)\gamma(b) = \gamma(a)$ whenever $a \subseteq b$.

For a cone γ , we denote by c_γ the apex of γ and the morphism $\gamma(a)$ is called the *component* of the cone γ at a . For a cone γ , define the set

$$m_\gamma := \{a : \gamma(a) \text{ is an isomorphism}\}.$$

We refer to the set m_γ as the *M-set* of the cone γ .

Definition 2.4. Let \mathcal{C} be a category with subobjects. A cone $\gamma : v\mathcal{C} \rightarrow \mathcal{C}$ is called *normal* if its *M-set* is non-empty, that is, there exists at least one $a \in v\mathcal{C}$ such that $\gamma(a) : a \rightarrow c_\gamma$ is an isomorphism.

Definition 2.5 ([23, Sec. III.1.3]). A category \mathcal{C} is said to be *normal* if:

- (NC 1) \mathcal{C} is a category with subobjects;
- (NC 2) every inclusion in \mathcal{C} splits;
- (NC 3) every morphism in \mathcal{C} admits a normal factorization;
- (NC 4) for each $c \in v\mathcal{C}$, there exists a normal cone μ such that $\mu(c) = 1_c$.

2.3. Normal category of a regular semigroup

As mentioned in the introduction, the concept of a normal category was devised by Nambooripad [23] to capture the structure of one-sided ideals of an arbitrary regular semigroup in purely categorical terms.

Recall that for a given regular semigroup S , the category $\mathbb{L}(S)$ of the principal left ideals of S has the set $\{Se : e \in E(S)\}$ as the set of objects and partial right translations (i.e. right translations restricted to a principal left ideal) as morphisms. Namely, for each pair of principal left ideals Se and Sf , the set of all morphisms from Se to Sf is $\{\rho(e, u, f) : u \in eSf\}$, where the map $\rho(e, u, f)$ sends $x \in Se$ to $xu \in Sf$. One can easily see that $\rho(e, u, f) = \rho(g, v, h)$ if and only if $e \mathcal{L} g, f \mathcal{L} h$ and $v = gu$ [23, Lemma III.12]. Two morphisms $\rho(e, u, f)$ and $\rho(g, v, h)$ are composable if and only if $Sf = Sg$ (i.e. if and only if $f \mathcal{L} g$) and then $\rho(e, u, f)\rho(g, v, h) = \rho(e, uv, h)$.

Define a subcategory $\mathcal{P}_{\mathbb{L}}$ of $\mathbb{L}(S)$ with $v\mathcal{P}_{\mathbb{L}} = v\mathbb{L}(S)$ and such that $\mathcal{P}_{\mathbb{L}}$ has a morphism from Se to Sf if and only if $Se \subseteq Sf$, in which case there is a unique morphism, namely $\rho(e, e, f)$. The morphisms of the subcategory $\mathcal{P}_{\mathbb{L}}$ correspond to the inclusions of principal ideals. By definition, $\mathcal{P}_{\mathbb{L}}$ is a strict pre-order. Clearly, every morphism $\rho(e, e, f)$ is a monomorphism. Also for morphisms $\rho(e, e, f), \rho(g, g, f) \in \mathcal{P}_{\mathbb{L}}$ such that $\rho(e, e, f) = \rho(h, u, g)\rho(g, g, f)$ in the category $\mathbb{L}(S)$, we have $\rho(e, e, f) = \rho(h, u, f)$, whence $u = he = h$. Therefore, the morphism $\rho(h, u, g) = \rho(h, h, g)$ belongs to $\mathcal{P}_{\mathbb{L}}$. Thus, all conditions of Definition 2.1 are satisfied and $(\mathbb{L}(S), \mathcal{P}_{\mathbb{L}})$ is a category with subobjects. Hence, the morphisms of $\mathbb{L}(S)$ belonging to $\mathcal{P}_{\mathbb{L}}$ are inclusions.

Now observe that for every inclusion $\rho(e, e, f)$, we have $fe \in fSe$ so that $\rho(f, fe, e)$ is a morphism in $\mathbb{L}(S)$. Since $Se \subseteq Sf$, we have $ef = e$ whence

$$\rho(e, e, f)\rho(f, fe, e) = \rho(e, e(fe), e) = \rho(e, (ef)e, e) = \rho(e, e, e) = 1_{Se}.$$

Thus, every inclusion in the category $\mathbb{L}(S)$ splits, and $\rho(f, fe, e)$ is a retraction.

It is shown in [23, Corollary III.14] that every morphism in the category $\mathbb{L}(S)$ admits a normal factorization in the sense of Definition 2.2.

Now, for each $a \in S$, we define a function $\rho^a : v\mathbb{L}(S) \rightarrow \mathbb{L}(S)$ as follows:

$$\rho^a(Se) := \rho(e, ea, f), \quad \text{where } f \mathcal{L} a. \tag{2.1}$$

It is easy to verify that the map ρ^a is well-defined, that is, it does not depend on the particular choice of an idempotent generator of the left ideal Se nor on the particular choice of an idempotent in the \mathcal{L} -class of a . Moreover, ρ^a is a normal cone with apex Sf in the sense of Definition 2.4, see [23, Lemma III.15].

In the sequel, the normal cone ρ^a is called the *principal cone* determined by the element a . In particular, observe that, for an idempotent $e \in E(S)$, we have a principal cone ρ^e such that $\rho^e(Se) = \rho(e, e, e) = 1_{Se}$.

Summarizing the above discussion, we see that the category $\mathbb{L}(S)$ satisfies all conditions (NC 1)–(NC 4) from the definition of a normal category (Definition 2.5).

2.4. Normal category of a locally inverse semigroup

Now, we aim to isolate categorical properties that distinguish locally inverse semigroups within the framework of Sec. 2.3.

Lemma 2.5. *If S is locally inverse, every inclusion $\rho(e, e, f) \in \mathbb{L}(S)$ splits uniquely, that is, if $\varphi, \psi \in \mathbb{L}(S)$ are such that $\rho(e, e, f)\varphi = \rho(e, e, f)\psi = 1_{Se}$, then $\varphi = \psi$.*

Proof. Take any element $g \in fSe$ such that the morphism $\rho(f, g, e)$ satisfies $\rho(e, e, f)\rho(f, g, e) = 1_{Se}$. Then $\rho(e, eg, e) = \rho(e, e, e)$ whence $eg = e$. As $g \in fSe$, we have $ge = g$ whence $g^2 = (ge)g = g(eg) = ge = g$ and $g \mathcal{L} e$. Besides that, we have $fg = gf = g$ since $g \in fSe$ and $ef = e$ in view of $Se \subseteq Sf$. We conclude that $g \in \omega(f)$. As S is locally inverse, Theorem 2.2(2) implies that each \mathcal{L} -class of S has at most one idempotent in common with $\omega(f)$ so that the conditions $g \mathcal{L} e$ and $g \in \omega(f)$ uniquely define g , and hence, the retraction $\rho(f, g, e)$. □

Remark 2.6. The proof of Lemma 2.5 shows that for an arbitrary retraction $\rho(f, g, e)$, the element g is an idempotent in $\omega(f)$ such that $g \mathcal{L} e$. Hence, $Se = Sg$ and so $\rho(f, g, e) = \rho(f, g, g)$ and $\rho(e, e, f) = \rho(g, g, f)$. We see that in $\mathbb{L}(S)$, every retraction may be represented as $\rho(f, g, g)$ and every inclusion may be represented as $\rho(g, g, f)$ for some $g \in \omega(f)$.

For an arbitrary regular semigroup S , morphisms of $\mathbb{L}(S)$ may admit several different normal factorizations. Our next lemma shows that locally inverse semigroups behave better in this respect.

Lemma 2.7. *If S is locally inverse, every morphism in $\mathbb{L}(S)$ has a unique normal factorization.*

Proof. Let $\rho(e, u, f)$ with $u \in eSf$ be a morphism in the category $\mathbb{L}(S)$. Consider an arbitrary normal factorization $\rho(e, u, f) = \theta\sigma\iota$, where θ is a retraction, σ is an isomorphism and ι is an inclusion in $\mathbb{L}(S)$. According to Remark 2.6, we can write $\theta = \rho(e, g, g)$ for some $g \in \omega(e)$ and $\iota = \rho(h, h, f)$ for some $h \in \omega(f)$. Then $\sigma = \rho(g, v, h)$ for some $v \in gSh$, and we have

$$\rho(e, u, f) = \theta\sigma\iota = \rho(e, g, g)\rho(g, v, h)\rho(h, h, f).$$

The right-hand side is equal to $\rho(e, gvh, f) = \rho(e, v, f)$ since $v \in gSh$, and we conclude that $u = v$. Thus, each normal factorization of $\rho(e, u, f)$ can be written as

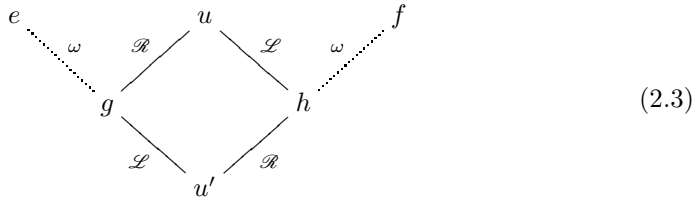
$$\rho(e, u, f) = \rho(e, g, g)\rho(g, u, h)\rho(h, h, f) \tag{2.2}$$

for some $g \in \omega(e)$ and some $h \in \omega(f)$.

Since $\rho(g, u, h)$ is an isomorphism in $\mathbb{L}(S)$, there exists $u' \in hSg$ such that

$$\begin{aligned} \rho(g, u, h)\rho(h, u', g) &= 1_{Sg} = \rho(g, g, g) \quad \text{and} \\ \rho(h, u', g)\rho(g, u, h) &= 1_{Sh} = \rho(h, h, h). \end{aligned}$$

This implies $uu' = g$ and $u'u = h$. Since $uu' = g$, we have $g \in uS$; on the other hand, as shown above $u = v = gvh$ whence $u \in gS$. We thus conclude that $g \mathcal{R} u$. Similarly, $h \mathcal{L} u$. The diagram (2.3) represents the relations between the elements e, f, g, h, u, u' .



Since S is locally inverse, Theorem 2.2(2) implies that each \mathcal{R} -class of S has at most one idempotent in common with $\omega(e)$ so that the conditions $g \mathcal{R} u$ and $g \in \omega(e)$ uniquely define the element g , and consequently fix the retraction $\rho(e, g, g)$. Similarly, the conditions $h \mathcal{L} u$ and $h \in \omega(f)$ uniquely define the idempotent h , and therefore, fix the inclusion $\rho(h, h, f)$. Moreover, as soon as g and h are fixed, so is the isomorphism $\rho(g, u, h)$. Altogether, the decomposition (2.2) is unique. □

Lemmas 2.5 and 2.7 show that for $\mathbb{L}(S)$ with S being locally inverse, the conditions (NC 2) and (NC 3) from Definition 2.5 hold with uniqueness. This motivates the next definition.

Definition 2.6. A category \mathcal{C} is said to be an *unambiguous category* if:

- (UC 1) \mathcal{C} is a category with subobjects;
- (UC 2) every inclusion in \mathcal{C} splits *uniquely*;
- (UC 3) every morphism in \mathcal{C} admits a *unique* normal factorization;
- (UC 4) for each $c \in v\mathcal{C}$ there exists a normal cone μ such that $\mu(c) = 1_c$.

Thus, an unambiguous category is a special version of a normal category wherein the axioms (NC 2) and (NC 3) are strengthened by (UC 2) and (UC 3), respectively. The following fact is straightforward.

Proposition 2.8. *For every locally inverse semigroup, its category of all principal left ideals is unambiguous.*

2.5. Semigroup of normal cones of an unambiguous category

Next, we proceed to show that every unambiguous category arises as $\mathbb{L}(S)$ for some locally inverse semigroup S . To this end, we use Nambooripad’s construction of the semigroup of normal cones of a normal category and show that it produces a locally inverse semigroup when being applied to an unambiguous category.

Given a normal category \mathcal{C} , we denote by $\widehat{\mathcal{C}}$ the set of all normal cones in \mathcal{C} . For $\gamma \in \widehat{\mathcal{C}}$, if $\varphi \in \mathcal{C}(c_\gamma, d)$ is an epimorphism, then as in [23, Lemma I.1], we can easily

see that the map

$$\gamma * \varphi : c \mapsto \gamma(c)\varphi \quad \text{for all } c \in v\mathcal{C} \tag{2.4}$$

is a normal cone such that $c_{\gamma*\varphi} = d$. Using this observation, one can define a binary composition on $\widehat{\mathcal{C}}$ as follows: for $\gamma, \delta \in \widehat{\mathcal{C}}$,

$$\gamma \cdot \delta := \gamma * (\delta(c_\gamma))^\circ, \tag{2.5}$$

where $(\delta(c_\gamma))^\circ$ is the epimorphic component of the morphism $\delta(c_\gamma)$.

Lemma 2.9 ([23, Theorem I.2]). *Let \mathcal{C} be a normal category. The set $\widehat{\mathcal{C}}$ of all its normal cones forms a regular semigroup under the binary composition (2.5). A normal cone μ in \mathcal{C} is an idempotent if and only if $\mu(c_\mu) = 1_{c_\mu}$.*

The following characterizations of the preorders \leq_ℓ, \leq_r and the order \leq in the semigroup $\widehat{\mathcal{C}}$ can be easily extracted from the discussion in [23, Sec. III.2].

Lemma 2.10. *Let μ, ν be normal cones in the semigroup $\widehat{\mathcal{C}}$. Then:*

- (1) $\nu \leq_\ell \mu$ if and only if $c_\nu \subseteq c_\mu$. In particular, $\nu \mathcal{L} \mu$ if and only if $c_\nu = c_\mu$.
- (2) $\nu \leq_r \mu$ if and only if $\nu(c_\mu)$ is an epimorphism such that $\nu = \mu * \nu(c_\mu)$. And $\nu \mathcal{R} \mu$ if and only if $\nu(c_\mu)$ is an isomorphism such that $\nu = \mu * \nu(c_\mu)$.

Lemma 2.11. *Let μ, ν be idempotents in the semigroup $\widehat{\mathcal{C}}$. Then $\nu \leq \mu$ if and only if $\nu(c_\mu)$ is a retraction such that $\nu = \mu * \nu(c_\mu)$.*

Proposition 2.12. *Let \mathcal{C} be an unambiguous category. The semigroup $\widehat{\mathcal{C}}$ of all normal cones in \mathcal{C} is locally inverse.*

Proof. By Lemma 2.9, $\widehat{\mathcal{C}}$ is a regular semigroup. So, it suffices to show that $\widehat{\mathcal{C}}$ satisfies condition (2) in Theorem 2.2. Thus, let μ, ν and ϑ be idempotent normal cones in \mathcal{C} with apices c, c_1 and c_2 , respectively, and $\nu, \vartheta \in \omega(\mu)$. By Lemma 2.11, there exist retractions $q_1 := \nu(c) : c \rightarrow c_1$ and $q_2 := \vartheta(c) : c \rightarrow c_2$ such that $\nu = \mu * q_1$ and $\vartheta = \mu * q_2$. We have to verify that $\nu = \vartheta$ whenever $\nu \mathcal{L} \cup \mathcal{R} \vartheta$.

First suppose that $\nu \mathcal{L} \vartheta$. Then $c_1 = c_2$ by Lemma 2.10(1). Since \mathcal{C} is an unambiguous category, the inclusion $\iota(c_1, c) = \iota(c_2, c)$ splits uniquely. So, the retraction between c and $c_1 = c_2$ must be unique, and thus, $q_1 = q_2$. Hence, $\nu = \mu * q_1 = \mu * q_2 = \vartheta$.

Now, let $\nu \mathcal{R} \vartheta$. By Lemma 2.10(2), we have $\nu = \vartheta * \nu(c_2)$. Equating the components at c , we have $\nu(c) = \vartheta(c)\nu(c_2)$, that is, $q_1 = q_2\nu(c_2)$. Dually, we have $q_2 = q_1\vartheta(c_1)$. Denote $u := \vartheta(c_1)$ and $v := \nu(c_2)$. Then, combining the equalities $q_1 = q_2v$ and $q_2 = q_1u$, we deduce $q_1 = q_1uv$. Multiplying through on the left by the inclusion $\iota(c_1, c)$, we get $\iota(c_1, c)q_1 = \iota(c_1, c)q_1uv$ whence $uv = 1_{c_1}$ since $\iota(c_1, c)q_1 = 1_{c_1}$ by the definition of a retraction. Dually, we obtain $vu = 1_{c_2}$. Thus, we see that u is an isomorphism.

Observe that each of the decompositions $q_2 = q_1u1_{c_2}$ and $q_2 = q_21_{c_2}1_{c_1}$ constitutes a normal factorization of the morphism q_2 . But since \mathcal{C} is an unambiguous

category, every morphism has a unique normal factorization. This implies that $q_1 = q_2$ also in this case. Hence, again $\nu = \mu * q_1 = \mu * q_2 = \vartheta$. \square

Two unambiguous categories are said to be isomorphic if there is an inclusion preserving isomorphism between them. Our next theorem follows by restricting [23, Theorem III.19] (which dealt with normal categories) to unambiguous categories.

Theorem 2.13. *Let \mathcal{C} be an unambiguous category and $\widehat{\mathcal{C}}$ its associated locally inverse semigroup of normal cones. Define a functor $F : \mathcal{C} \rightarrow \mathbb{L}(\widehat{\mathcal{C}})$ as follows:*

$$vF(c) := \widehat{\mathcal{C}}\mu \quad \text{for } c \in v\mathcal{C},$$

$$F(f) := \rho(\mu, \mu * f^\circ, \nu) \quad \text{for } f \in \mathcal{C}(c, d),$$

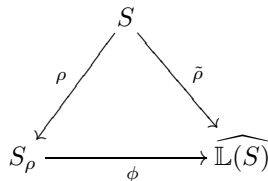
where $\mu, \nu \in E(\widehat{\mathcal{C}})$ are such that $c_\mu = c$ and $c_\nu = d$. Then F is an isomorphism of unambiguous categories.

Proposition 2.8 shows that the category $\mathbb{L}(S)$ of a locally inverse semigroup S is unambiguous. Conversely, given an abstract unambiguous category \mathcal{C} , Proposition 2.12 shows that its semigroup $\widehat{\mathcal{C}}$ is locally inverse, and by Theorem 2.13, the unambiguous category $\mathbb{L}(\widehat{\mathcal{C}})$ is isomorphic to \mathcal{C} . That is, every unambiguous category arises as the category $\mathbb{L}(S)$ of some locally inverse semigroup S . Thus, we arrive at the following corollary that completely characterizes the category of principal left ideals of a locally inverse semigroup.

Corollary 2.14. *A category is isomorphic to the category $\mathbb{L}(S)$ for some locally inverse semigroup S if and only if it is unambiguous.*

Recall from [11] that the right regular representation of a semigroup S is the homomorphism $\rho : a \mapsto \rho_a$ of S into the full transformation semigroup on the set S . Denote the image of ρ by S_ρ ; then $\rho : S \rightarrow S_\rho$ is a surjective homomorphism. The next proposition follows directly from [23, Theorem III.16].

Proposition 2.15. *Let S be a locally inverse semigroup. The map $a \mapsto \rho^a$ (where ρ^a is the principal cone determined by a) defines a homomorphism $\tilde{\rho} : S \rightarrow \widehat{\mathbb{L}(S)}$. Also the map $\rho_a \mapsto \rho^a$ defines an injective homomorphism $\phi : S_\rho \rightarrow \widehat{\mathbb{L}(S)}$ such that the next diagram commutes.*



In particular, S is isomorphic to a subsemigroup of $\widehat{\mathbb{L}(S)}$ via $\tilde{\rho}$ if and only if ρ is injective.

Remark 2.16. Dually, we can define the unambiguous category $\mathbb{R}(S)$ of principal right ideals of a locally inverse semigroup S as follows:

$$v\mathbb{R}(S) := \{eS : e \in E(S)\},$$

$$\mathbb{R}(S)(eS, fS) := \{\lambda(e, u, f) : u \in fSe\},$$

where the partial left translation $\lambda(e, u, f)$ sends $x \in eS$ to $ux \in fS$. All notions defined for the category $\mathbb{L}(S)$ have their natural ‘duals’ in the category $\mathbb{R}(S)$; in particular, λ^a stands for the principal cone in $\mathbb{R}(S)$ determined by $a \in S$.

2.6. Unambiguous dual and cross-connections

We have seen in the previous subsections that given a locally inverse semigroup S , the categories $\mathbb{L}(S)$ and $\mathbb{R}(S)$ are unambiguous categories. In this subsection, we address the converse question: given two unambiguous categories \mathcal{C} and \mathcal{D} , whether a locally inverse semigroup S can be constructed such that \mathcal{C} and \mathcal{D} are isomorphic to $\mathbb{L}(S)$ and $\mathbb{R}(S)$, respectively. To answer this question, we first need to capture the relationship between the unambiguous categories $\mathbb{L}(S)$ and $\mathbb{R}(S)$, in a categorical framework. This is done via the notion of cross-connection which describes the relationship between the categories $\mathbb{L}(S)$ and $\mathbb{R}(S)$ using a pair of functors.

To this end, first recall that for a given normal category \mathcal{C} , its normal dual \mathcal{C}^* is defined as a full subcategory of the functor category $[\mathcal{C}, \mathbf{Set}]$ such that the objects of \mathcal{C}^* are certain special set valued functors called H -functors.

Let μ be an idempotent cone in a normal category \mathcal{C} . Then we define an H -functor $H(\mu; -) : \mathcal{C} \rightarrow \mathbf{Set}$ as follows: for $c \in v\mathcal{C}$ and $g \in \mathcal{C}(c, d)$,

$$H(\mu; c) := \{\mu * f^\circ : f \in \mathcal{C}(c_\mu, c)\},$$

$$H(\mu; g) : H(\mu; c) \rightarrow H(\mu; d) \text{ is given by } \mu * f^\circ \mapsto \mu * (fg)^\circ.$$
(2.6)

It can be shown [23, Lemma III.6] that the functor $H(\mu; -)$ is representable, that is, there exists an associated natural isomorphism $\eta_\mu : H(\mu; -) \rightarrow \mathcal{C}(c_\mu, -)$, where $\mathcal{C}(c_\mu, -)$ is the covariant hom-functor determined by the object c_μ .

Now, we proceed to characterize the normal dual associated with an unambiguous category. Given an unambiguous category \mathcal{C} , we define the *unambiguous dual* \mathcal{C}^* (often referred to as just *dual* below) as the full subcategory of $[\mathcal{C}, \mathbf{Set}]$ such that

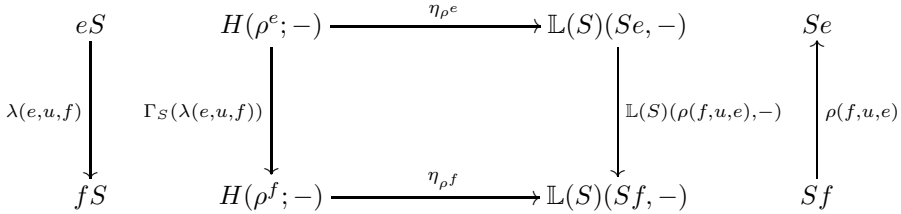
$$v\mathcal{C}^* := \{H(\mu; -) : \mu \in E(\widehat{\mathcal{C}})\}.$$

Hence, the morphisms in \mathcal{C}^* are natural transformations between the H -functors. Then [23, Theorem III.25] leads us to the following theorem.

Theorem 2.17. *Let \mathcal{C} be an unambiguous category with the dual \mathcal{C}^* . The unambiguous category $\mathbb{R}(\widehat{\mathcal{C}})$ is isomorphic to \mathcal{C}^* . In particular, the unambiguous dual \mathcal{C}^* is also an unambiguous category.*

Now, we proceed to describe how the unambiguous categories $\mathbb{L}(S)$ and $\mathbb{R}(S)$ arising from the same locally inverse semigroup S are interrelated. This interrelation is captured by a pair of functors Γ_S and Δ_S . The functor $\Gamma_S : \mathbb{R}(S) \rightarrow \mathbb{L}(S)^*$ is defined as follows. For each $eS \in v\mathbb{R}(S)$ and for each morphism $\lambda(e, u, f) \in \mathbb{R}(S)$, let η_{ρ^e} and η_{ρ^f} be the natural isomorphisms associated with the H -functors $H(\rho^e; -)$ and $H(\rho^f; -)$, respectively; then as illustrated in the diagram below:

$$v\Gamma_S(eS) := H(\rho^e; -) \quad \text{and} \quad \Gamma_S(\lambda(e, u, f)) := \eta_{\rho^e} \mathbb{L}(S)(\rho(f, u, e), -) \eta_{\rho^f}^{-1}. \quad (2.7)$$



Similarly, we define $\Delta_S : \mathbb{L}(S) \rightarrow \mathbb{R}(S)^*$ as follows: for each $Se \in v\mathbb{L}(S)$ and for each morphism $\rho(e, u, f) \in \mathbb{L}(S)$,

$$v\Delta_S(Se) := H(\lambda^e; -) \quad \text{and} \quad \Delta_S(\rho(e, u, f)) := \eta_{\lambda^e} \mathbb{R}(S)(\lambda(f, u, e), -) \eta_{\lambda^f}^{-1}. \quad (2.8)$$

As in [23, Theorem IV.2], we can prove that Γ_S and Δ_S are well defined covariant functors which are inclusion preserving and fully faithful. Moreover, for each $eS \in v\mathbb{R}(S)$, the restriction of Γ_S to the full subcategory of $\mathbb{R}(S)$ whose objects are the principal right ideals contained in eS is an isomorphism; a similar property holds for Δ_S and each $Se \in v\mathbb{L}(S)$.

The latter observation motivates the notion of a local isomorphism. An *ideal* (c) of an unambiguous category \mathcal{C} is the full subcategory of \mathcal{C} with objects

$$v(c) := \{d \in v\mathcal{C} : d \subseteq c\}.$$

Definition 2.7. A functor F between two unambiguous categories \mathcal{C} and \mathcal{D} is said to be a *local isomorphism* if F is inclusion preserving, fully faithful and for each $c \in v\mathcal{C}$, $F|_{v(c)}$ is an isomorphism of the ideal (c) onto $(F(c))$.

Thus, Γ_S and Δ_S are local isomorphisms. It must be noted here that the local isomorphism Γ_S as defined above are isomorphisms between (eS) and $(\Gamma_S(eS)) \subsetneq \mathbb{L}(S)^*$, for each $eS \in v\mathbb{R}(S)$; in general, the image ideal $(\Gamma_S(eS))$ is a proper ideal of the category $\mathbb{L}(S)^*$.

To describe the relationship between the local isomorphisms Γ_S and Δ_S , we need to employ the notion of the M -set $MH(\mu; -)$ of an H -functor $H(\mu; -)$ in an unambiguous category \mathcal{C} . It is defined as follows:

$$MH(\mu; -) := \{c \in v\mathcal{C} : \mu(c) \text{ is an isomorphism}\}. \quad (2.9)$$

So, the M -set $M\Gamma_S(fS)$ is $MH(\rho^f; -)$ according to (2.7); similarly, $M\Delta_S(Se) = MH(\lambda^e; -)$ by (2.8). Now, it can be seen that for objects $Se \in v\mathbb{L}(S)$ and $fS \in v\mathbb{R}(S)$,

$$Se \in M\Gamma_S(fS) \text{ if and only if } fS \in M\Delta_S(Se). \tag{2.10}$$

The above statement leads us to the definition of a cross-connection.

Definition 2.8. Let \mathcal{C} and \mathcal{D} be unambiguous categories. A *cross-connection* between \mathcal{C} and \mathcal{D} is a quadruplet $(\mathcal{C}, \mathcal{D}; \Gamma, \Delta)$, where $\Gamma : \mathcal{D} \rightarrow \mathcal{C}^*$ and $\Delta : \mathcal{C} \rightarrow \mathcal{D}^*$ are local isomorphisms such that for $c \in v\mathcal{C}$ and $d \in v\mathcal{D}$,

$$c \in M\Gamma(d) \Leftrightarrow d \in M\Delta(c). \tag{2.11}$$

Summarizing the above discussion, we have proved the following theorem.

Theorem 2.18. *Let S be a locally inverse semigroup with unambiguous categories $\mathbb{L}(S)$ and $\mathbb{R}(S)$. Define functors Γ_S and Δ_S as in (2.7) and (2.8). Then $\Omega_S = (\mathbb{L}(S), \mathbb{R}(S); \Gamma_S, \Delta_S)$ is a cross-connection between $\mathbb{L}(S)$ and $\mathbb{R}(S)$.*

2.7. Locally inverse semigroup of a cross-connection

We have shown how a locally inverse semigroup gives rise to a cross-connection of two unambiguous categories. Now, we describe the converse construction. We build the locally inverse semigroup arising from a cross-connection between two unambiguous categories. Recall from Sec. 2.5 that given an unambiguous category, we have an associated locally inverse semigroup. We will be identifying the required locally inverse semigroup associated with a cross-connection as a subdirect product of the locally inverse semigroups arising from the two unambiguous categories, i.e. as a semigroup of pairs of normal cones which ‘respect’ the cross-connection.

First, observe that given a cross-connection $\Omega = (\mathcal{C}, \mathcal{D}; \Gamma, \Delta)$, by a well-known category isomorphism [18]

$$[\mathcal{C}, [\mathcal{D}, \mathbf{Set}]] \cong [\mathcal{C} \times \mathcal{D}, \mathbf{Set}],$$

we obtain two bifunctors $\Gamma(-, -)$ and $\Delta(-, -)$ from $\mathcal{C} \times \mathcal{D}$ to \mathbf{Set} .

Now, for the cross-connection $\Omega = (\mathcal{C}, \mathcal{D}; \Gamma, \Delta)$, the set

$$E_\Omega = \{(c, d) \in v\mathcal{C} \times v\mathcal{D} : c \in M\Gamma(d)\} \tag{2.12}$$

is the regular biordered set associated with the cross-connection Ω [23]. We show later that the set E_Ω is in fact a pseudo-semilattice.

Recall that for each $(c, d) \in E_\Omega$, there is a uniquely defined idempotent cone $\gamma(c, d)$ in the unambiguous category \mathcal{C} such that

$$c_{\gamma(c,d)} = c \text{ and } H(\gamma(c, d); -) = \Gamma(d). \tag{2.13}$$

Similarly, for each pair $(c, d) \in E_\Omega$, we have a unique idempotent cone $\delta(c, d)$ in \mathcal{D} such that

$$c_{\delta(c,d)} = d \text{ and } H(\delta(c, d); -) = \Delta(c). \tag{2.14}$$

Given a cross-connection $\Omega = (\mathcal{C}, \mathcal{D}; \Gamma, \Delta)$, $(c', d), (c, d') \in E_\Omega$, $f \in \mathcal{C}(c', c)$ and $g \in \mathcal{D}(d', d)$, the morphism g is called the *transpose* of f if the morphisms f and g make the following diagram commute.

$$\begin{array}{ccccc}
 c' & & \Delta(c') & \xrightarrow{\eta_{\delta(c',d)}} & \mathcal{D}(d, -) & & d \\
 \downarrow f & & \downarrow \Delta(f) & & \downarrow \mathcal{D}(g, -) & & \uparrow g \\
 c & & \Delta(c) & \xrightarrow{\eta_{\delta(c,d')}} & \mathcal{D}(d', -) & & d'
 \end{array}$$

The transpose $g \in \mathcal{D}(d', d)$ is unique for a given pair of elements in E_Ω . The transpose of $f \in \mathcal{C}(c', c)$ will be denoted by f^\dagger in the sequel. Then we have the following theorem which is a consequence of [23, Theorem IV.16], in the notation introduced above.

Theorem 2.19. *Given unambiguous categories \mathcal{C} and \mathcal{D} and a cross-connection $\Omega = (\mathcal{C}, \mathcal{D}; \Gamma, \Delta)$ with associated bifunctors $\Gamma(-, -)$ and $\Delta(-, -)$ for each $(c, d) \in v\mathcal{C} \times v\mathcal{D}$, the map $\chi(c, d) : \Gamma(c, d) \rightarrow \Delta(c, d)$ given by*

$$\chi(c, d) : \gamma(c', d) * f^\circ \mapsto \delta(c, d') * (f^\dagger)^\circ$$

is a bijection, where $c' \in M\Gamma(d)$ and $d' \in M\Delta(c)$ and $f^\dagger \in \mathcal{D}(d', d)$ is the transpose of the morphism $f \in \mathcal{C}(c', c)$. Also the map $(c, d) \mapsto \chi(c, d)$ defines a natural isomorphism between the bifunctors $\Gamma(-, -)$ and $\Delta(-, -)$.

Now, as in [23, Sec. IV.5.1], we obtain the following regular subsemigroups of the semigroups $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{D}}$ from the bifunctors $\Gamma(-, -)$ and $\Delta(-, -)$, respectively,

$$\widehat{\Gamma} = \bigcup \{ \Gamma(c, d) : (c, d) \in v\mathcal{C} \times v\mathcal{D} \}, \tag{2.15a}$$

$$\widehat{\Delta} = \bigcup \{ \Delta(c, d) : (c, d) \in v\mathcal{C} \times v\mathcal{D} \}. \tag{2.15b}$$

Since $\widehat{\Gamma}$ and $\widehat{\Delta}$ are regular subsemigroups of the locally inverse semigroups $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{D}}$, respectively, $\widehat{\Gamma}$ and $\widehat{\Delta}$ are locally inverse semigroups.

Then we can see that a normal cone γ belongs to $\widehat{\Gamma}$ if and only if $\gamma = \gamma(c_1, d_1) * u$, where u is an isomorphism in \mathcal{C} and $(c_1, d_1) \in E_\Omega$. Dually, a normal cone δ belongs to $\widehat{\Delta}$ if and only if $\delta = \delta(c_1, d_1) * u$, where u is an isomorphism in \mathcal{D} .

Now, we proceed to build the cross-connection semigroup associated with the cross-connection as a subdirect product of the locally inverse semigroups $\widehat{\Gamma}$ and $\widehat{\Delta}$. Recall that χ as defined in Theorem 2.19 is a natural isomorphism between the bifunctors $\Gamma(-, -)$ and $\Delta(-, -)$. This natural isomorphism gives rise to a ‘linking’ between the locally inverse semigroups $\widehat{\Gamma}$ and $\widehat{\Delta}$.

Definition 2.9. Given a cross-connection $\Omega = (\Gamma, \Delta; \mathcal{C}, \mathcal{D})$, a normal cone $\gamma \in \widehat{\Gamma}$ is said to be *linked* to $\delta \in \widehat{\Delta}$ if there is a $(c, d) \in v\mathcal{C} \times v\mathcal{D}$ such that $\gamma \in \Gamma(c, d)$ and $\delta = \chi(c, d)(\gamma)$; we then say that the pair (γ, δ) is a *linked pair*.

Given a cross-connection $\Omega = (\Gamma, \Delta; \mathcal{C}, \mathcal{D})$ of unambiguous categories \mathcal{C} and \mathcal{D} , define the set

$$\mathbb{S}\Omega = \{(\gamma, \delta) \in \widehat{\Gamma} \times \widehat{\Delta} : (\gamma, \delta) \text{ is linked}\}. \tag{2.16}$$

Define an operation on $\mathbb{S}\Omega$ as follows:

$$(\gamma, \delta) \circ (\gamma', \delta') = (\gamma \cdot \gamma', \delta' \cdot \delta) \quad \text{for all } (\gamma, \delta), (\gamma', \delta') \in \mathbb{S}\Omega.$$

Suppose $(\gamma, \delta), (\gamma', \delta') \in \mathbb{S}\Omega$, then as in the [23, Lemma IV.30], we can show that $\gamma \cdot \gamma'$ is linked to $\delta' \cdot \delta$. Further by [23, Theorem IV.32], we see that $\mathbb{S}\Omega$ is a regular semigroup called the *cross-connection semigroup* determined by Ω . Then the set of idempotents of the semigroup $\mathbb{S}\Omega$ is given by

$$E(\mathbb{S}\Omega) = \{(\gamma(c, d), \delta(c, d)) : (c, d) \in E_\Omega\}.$$

Since $\mathbb{S}\Omega$ is a regular semigroup, the set $E(\mathbb{S}\Omega)$ is a regular biordered set. By the discussion in [23, Sec. V.1.2], we can see that the set $E(\mathbb{S}\Omega)$ is regular biorder isomorphic with the set E_Ω under the map

$$(\gamma(c, d), \delta(c, d)) \mapsto (c, d).$$

More precisely, as in [23, Sec. V.1.2], we can show that the quasi orders in the set $E(\mathbb{S}\Omega) = E_\Omega$ are given by

$$(c, d) \leq_\ell (c', d') \Leftrightarrow c \subseteq c' \quad \text{and} \quad (c, d) \leq_r (c', d') \Leftrightarrow d \subseteq d'$$

so that E_Ω forms a regular biordered set with the basic products and sandwich sets as described in [23, Sec. V.1.2].

Theorem 2.20. *Given a cross-connection $\Omega = (\Gamma, \Delta; \mathcal{C}, \mathcal{D})$ of unambiguous categories \mathcal{C} and \mathcal{D} , the cross-connection semigroup $\mathbb{S}\Omega$ is locally inverse.*

Proof. Observe that the cross-connection semigroup $\mathbb{S}\Omega$ is a subdirect product of two locally inverse semigroups $\widehat{\Gamma}$ and $\widehat{\Delta}$. It is well-known that a regular subdirect product of two locally inverse semigroups is locally inverse. Hence, the theorem. \square

Corollary 2.21. *Given a cross-connection $\Omega = (\Gamma, \Delta; \mathcal{C}, \mathcal{D})$ of unambiguous categories \mathcal{C} and \mathcal{D} , the biordered set E_Ω is a pseudo-semilattice.*

Further, by [23, Theorem IV.35], we have the following theorem.

Theorem 2.22. *For a cross-connection $\Omega = (\Gamma, \Delta; \mathcal{C}, \mathcal{D})$ with the cross-connection semigroup $\mathbb{S}\Omega$, the unambiguous categories $\mathbb{L}(\mathbb{S}\Omega)$ and $\mathbb{R}(\mathbb{S}\Omega)$ are isomorphic to the categories \mathcal{C} and \mathcal{D} , respectively.*

It is known that the category **LIS** of locally inverse semigroups forms a full subcategory of the category **RS** of regular semigroups. Similarly, we can see that the category **CUC** of cross-connections of unambiguous categories forms a full subcategory of the category **Cr** of cross-connections of normal categories. Hence,

by [23, Theorem V.18], we obtain the following structure theorem for locally inverse semigroups.

Theorem 2.23. *The category LIS of locally inverse semigroups is equivalent to the category CUC of cross-connections of unambiguous categories.*

3. The Inverse Case

In this section, we specialize the results of Sec. 2 to obtain a new structure theorem for inverse semigroups. We will see that the full machinery of cross-connections is not required for this task due to the intrinsic symmetry of inverse semigroups. It also turns out that normal cones, our previous building blocks, are too general to build inverse semigroups. We replace them with certain subspecies called inversive cones. As the reader will see, employing these cones, we obtain the structure theorem for inverse semigroups using a single category. We characterize this category as an inversive category and prove a category equivalence between the category IS of inverse semigroups and the category IC of inversive categories; as a parallel to the Ehresmann–Schein–Nambooripad Theorem (which describes a category isomorphism between inverse semigroups and inductive groupoids). Further, we will also outline how an inductive groupoid is ‘sitting inside’ a given inversive category; thereby describing the equivalence of these approaches. Thus, we are able to partly recover the Ehresmann–Schein–Nambooripad Theorem from our considerations of locally inverse semigroups.

3.1. Inverse semigroups and inversive categories

We begin by analyzing the category $\mathbb{L}(S)$, where S is an inverse semigroup. Our discussion will develop as follows: registering certain properties^a of $\mathbb{L}(S)$ is interwoven with introducing appropriate abstract notions that capture these properties. Eventually, we arrive at a collection of notions that provides a complete abstract characterization for categories of the form $\mathbb{L}(S)$ with S being an inverse semigroup.

As in Sec. 2, we observe that the category $\mathbb{L}(S)$ is a category with subobjects. Since S is inverse, the set $E(S)$ forms a semilattice. So, the set $v\mathbb{L}(S)$ along with the partial order \leq defined by

$$Se \leq Sf \Leftrightarrow Se \subseteq Sf$$

also forms a semilattice. Abstracting this, we have the following definition.

Definition 3.1. A category with subobjects $(\mathcal{C}, \mathcal{P})$ is called a *semilattice ordered category* (abbreviated as *so-category* in the sequel) if $v\mathcal{P}$ forms a semilattice with

^aSome of them had been discussed in [29], but we reprove these properties here in our notation for the sake of completeness and uniformity.

respect to the relation \leq defined as

$$p \leq q \Leftrightarrow \text{there is an inclusion from } p \text{ to } q$$

for any $p, q \in v\mathcal{P}$.

Given any two objects c, c' in an so-category \mathcal{C} , there exists a unique object d in \mathcal{C} such that (1) there are inclusions from d to c and c' , (2) for every object d' such that there are inclusions from d' to c and c' , there is an inclusion from d' to d . Hence, this ‘maximal’ object d acts as the *meet* of the objects c and c' . We denote the unique object d in $v\mathcal{C}$ by $c \wedge c'$.

When S is an inverse semigroup, we can easily see that any inclusion in $\mathbb{L}(S)$ will be of the form $\rho(e, e, f)$ and any retraction will be of the form $\rho(f, e, e)$, where $e, f \in E(S)$ with $e \leq f$. Any isomorphism in $\mathbb{L}(S)$ will be of the form $\rho(aa^{-1}, a, a^{-1}a)$ for some $a \in S$; equivalently of the form $\rho(e, u, f)$ for some $e, f \in E(S)$ and some $u \in S$ such that $e \mathcal{R} u \mathcal{L} f$. For an arbitrary morphism $\rho(e, u, f) \in \mathbb{L}(S)$, we can easily see that its unique normal factorization is given by

$$\rho(e, u, f) = \rho(e, g, g)\rho(g, u, h)\rho(h, h, f),$$

where $g = uu^{-1}$ and $h = u^{-1}u$. Also, for an inclusion $\rho(e, e, f)$, its unique retraction is given by $\rho(f, e, e)$.

Observe that in an so-category \mathcal{C} where every inclusion splits uniquely, we have two morphisms uniquely associated with any pair $c, c' \in v\mathcal{C}$ such that $c \leq c'$, namely, the inclusion $\iota(c, c')$ and its retraction $q(c', c)$. The subcategory generated by the retractions and inclusions in \mathcal{C} is called the *core* of \mathcal{C} and denoted by $\langle \mathcal{C} \rangle$.

When S is an inverse semigroup, any morphism ρ in the core category $\langle \mathbb{L}(S) \rangle$ can be written as $\rho = \rho(e_1, e_1, e_2)\rho(e_2, e_3, e_3) \cdots \rho(e_{2n-1}, e_{2n-1}, e_{2n})\rho(e_{2n}, e_{2n}, e_{2n+1})$ where each ‘odd’ factor $\rho(e_{2i-1}, e_{2i-1}, e_{2i})$ is an inclusion and each ‘even’ factor $\rho(e_{2i}, e_{2i}, e_{2i+1})$ is a retraction. (To ensure that the factorization of ρ starts with an inclusion and ends with a retraction, we can, if necessary, prepend or append a morphism of the form $1_e = \rho(e, e, e)$ which serves both as an inclusion and a retraction). Then $\rho = \rho(e_1, e_1 e_3 \cdots e_{2n+1}, e_{2n+1})$. Since $E(S)$ is a semilattice, we see that the element $e_1 e_3 \cdots e_{2n+1}$ is an idempotent, say f , and so the unique normal factorization of ρ will be given by

$$\rho = \rho(e_1, f, f)\rho(f, f, e_{2n+1}),$$

where $\rho(e_1, f, f)$ is a retraction and $\rho(f, f, e_{2n+1})$ is an inclusion. This leads to the following definition of a special kind of normal factorization.

Definition 3.2. Let \mathcal{C} be an so-category in which inclusions split uniquely. A morphism $\varphi = \iota(c_1, c_2)q(c_2, c_3) \cdots \iota(c_{2n-1}, c_{2n})q(c_{2n}, c_{2n+1})$ in the core $\langle \mathcal{C} \rangle$ is said to have an *inversive factorization* if $\varphi = q(c_1, d)\iota(d, c_{2n+1})$, where $q(c_1, d)$ is the retraction from c_1 to $d := \bigwedge_{i=0}^n c_{2i+1}$ and $\iota(d, c_{2n+1})$ is the inclusion from d to c_{2n+1} .

Remark 3.1. Let \mathcal{C} be an so-category with inclusions splitting uniquely where every morphism in the core $\langle \mathcal{C} \rangle$ has an inversive factorization. Then if $\varphi : c \rightarrow d$ is an isomorphism in $\langle \mathcal{C} \rangle$, then $c = d$ and $\varphi = 1_c$, that is, the only isomorphisms in $\langle \mathcal{C} \rangle$ are the identity morphisms.

As mentioned earlier, normal cones prove to be too general for the inverse case. So, we proceed to analyze the principal cone ρ^a in an inverse semigroup S to characterize our apex building blocks. Recall from Eq. (2.1) that the principal cone $\rho^a : v\mathbb{L}(S) \rightarrow \mathbb{L}(S)$ with apex $c_\gamma = Sa = Sf$ is defined for each $Se \in v\mathbb{L}(S)$ as $\rho^a(Se) = \rho(e, ea, f)$, where $f \in E(L_a)$. Since S is inverse, there is a unique idempotent in the \mathcal{L} -class of the element a so that $f = a^{-1}a$. Now, recall from [23, Lemma III.15] that the set

$$m_{\rho^a} := \{Se : \rho^a(Se) \text{ is an isomorphism}\} = \{Se : e \in E(R_a)\} = \{Saa^{-1}\}.$$

That is, for a principal cone ρ^a in the category $\mathbb{L}(S)$, where S is an inverse semigroup, there is a unique isomorphism component and so the M -set m_{ρ^a} is a singleton. In what follows, we use the same notation for this singleton set and its unique element.

Let ρ^a be a principal cone in the category $\mathbb{L}(S)$ and let $Sf := Sa^{-1}a$ be its apex and $Sg := Saa^{-1}$ its M -set. Then for an arbitrary object $Se \in v\mathbb{L}(S)$, we have

$$\begin{aligned} \rho^a(Se) &= \rho(e, ea, f) \\ &= \rho(e, eg, eg)\rho(eg, ea, h)\rho(h, h, f) \quad \text{using normal factorization} \\ &\qquad\qquad\qquad \text{and } h = a^{-1}ea \\ &= \rho(e, eg, eg)\rho(eg, ea, f) \quad \text{since } eah = ea(a^{-1}ea) = egea = ega \\ &= q(Se, Seg)\rho^a(Seg) \\ &= q(Se, Seg)\iota(Seg, Sg)\rho^a(Sg) \quad \text{since } \rho^a \text{ is a cone and } Seg \subseteq Sg. \end{aligned}$$

That is, the component of the principal cone ρ^a at any object Se is composed of:

- (i) the retraction from Se to the object $Seg = Se \wedge m_{\rho^a}$,
- (ii) the inclusion from $Se \wedge m_{\rho^a}$ to the object m_{ρ^a} ,
- (iii) the component of the principal cone at m_{ρ^a} .

Hence, $\text{coim } \rho^a(Se) = Se \wedge m_{\rho^a}$. The above discussion inspires us to isolate certain special normal cones in an so-category, which we call inversive cones.

Definition 3.3. A cone γ in an unambiguous so-category \mathcal{C} is said to be an *inversive cone* if:

- (1) the M -set m_γ is a singleton,
- (2) for each $c \in v\mathcal{C}$, $\text{coim } \gamma(c) = c \wedge m_\gamma$.

Lemma 3.2. *Let \mathcal{C} be an unambiguous so-category. Then for any inversive cone γ and any $c \in v\mathcal{C}$,*

$$\gamma(c) = q(c, c \wedge m_\gamma)\gamma(c \wedge m_\gamma).$$

Proof. For any $c \in v\mathcal{C}$, since $\text{coim } \gamma(c) = c \wedge m_\gamma$, the unique normal factorization of the morphism $\gamma(c)$ has the form $\gamma(c) = q(c, c \wedge m_\gamma)uj$ for some isomorphism u and inclusion j . Then by the definition of a cone (Definition 2.3), we have

$$\gamma(c \wedge m_\gamma) = \iota(c \wedge m_\gamma, c)\gamma(c) = \iota(c \wedge m_\gamma, c)q(c, c \wedge m_\gamma)uj = uj,$$

since $\iota(c \wedge m_\gamma, c)q(c, c \wedge m_\gamma) = 1_{c \wedge m_\gamma}$ by the definition of a retraction. Hence, $\gamma(c) = q(c, c \wedge m_\gamma)\gamma(c \wedge m_\gamma)$ for any $c \in v\mathcal{C}$. □

Remark 3.3. Observe that an inversive cone γ in an unambiguous so-category \mathcal{C} gets completely determined by the component of γ at the object $m_\gamma \in v\mathcal{C}$. Indeed, for any $c \in v\mathcal{C}$, we have $\gamma(c) = q(c, c \wedge m_\gamma)\gamma(c \wedge m_\gamma)$ by Lemma 3.2. But since $c \wedge m_\gamma \leq m_\gamma$, we have $\gamma(c \wedge m_\gamma) = \iota(c \wedge m_\gamma, m_\gamma)\gamma(m_\gamma)$ by the definition of a cone. Hence, $\gamma(c) = q(c, c \wedge m_\gamma)\iota(c \wedge m_\gamma, m_\gamma)\gamma(m_\gamma)$. This leads to the following lemma.

Lemma 3.4. *If S is an inverse semigroup, for every $Sf \in \mathbb{L}(S)$, there exists a unique idempotent inversive cone with apex Sf , namely, ρ^f .*

Proof. Clearly, the cone ρ^f is an idempotent inversive cone with apex Sf . Let μ be an arbitrary idempotent inversive cone with apex Sf . Since μ is an idempotent cone, by Lemma 2.9, we have $\mu(Sf) = 1_{Sf}$. But since μ is an inversive cone, the M -set m_μ is a singleton and this implies that $m_\mu = \{Sf\}$. Also, for an arbitrary $Se \in v\mathbb{L}(S)$, we have $\text{coim } \mu(Se) = Se \wedge m_\mu$. So,

$$\begin{aligned} \mu(Se) &= q(Se, Se \wedge m_\mu)\mu(Se \wedge m_\mu) \\ &= q(Se, Se \wedge m_\mu)\iota(Se \wedge m_\mu, m_\mu)\mu(m_\mu) \quad \text{since } Se \wedge m_\mu \leq m_\mu \\ &= \rho(e, ef, ef)\rho(ef, ef, f)\rho(f, f, f) \quad \text{since } m_\mu = \{Sf\} \\ &= \rho(e, ef, f) \\ &= \rho^f(Se). \end{aligned}$$

So $\mu = \rho^f$ and hence the lemma. □

Now, we collect all properties of the category of the principal left ideals of an inverse semigroup registered so far into a suitable categorical abstraction.

Definition 3.4. A category \mathcal{C} is said to be an *inversive category* if:

- (IC 1) \mathcal{C} is an so-category;
- (IC 2) every inclusion in \mathcal{C} splits *uniquely*;
- (IC 3) every morphism in \mathcal{C} admits a *unique* normal factorization;

- (IC 4) every morphism in the core $\langle \mathcal{C} \rangle$ has an *inversive* factorization;
- (IC 5) for each $c \in v\mathcal{C}$, there is a *unique inversive* idempotent cone with apex c .

Observe that an inversive category is indeed unambiguous. In the sequel, given an object c in an inversive category \mathcal{C} , the unique inversive idempotent cone with apex c is denoted by μ_c .

An inclusion preserving functor F between two inversive categories \mathcal{C}_1 and \mathcal{C}_2 is called an *inversive functor* if for any two objects $c, c' \in v\mathcal{C}_1$,

$$vF(c \wedge c') = vF(c) \wedge vF(c').$$

It is easy to see that inversive categories with inversive functors as morphisms form a locally small category **IC**.

We have already seen that given an inverse semigroup S , the category $\mathbb{L}(S)$ is inversive. Further, given a homomorphism ϕ between two inverse semigroups S_1 and S_2 , we can define a functor $\Phi : \mathbb{L}(S_1) \rightarrow \mathbb{L}(S_2)$ as follows. For idempotents $e, f \in S_1$ and $u \in eS_1f$,

$$v\Phi : S_1e \mapsto S_2e\phi; \quad \Phi : \rho(e, u, f) \mapsto \rho(e\phi, u\phi, f\phi). \tag{3.1}$$

It is easy to verify that Φ is an inversive functor. We have the following proposition.

Proposition 3.5. *The assignment*

$$S \mapsto \mathbb{L}(S); \quad \phi \mapsto \Phi$$

*constitutes a functor, say \mathbf{C} , from the category **IS** of inverse semigroups to the category **IC** of inversive categories.*

3.2. Inverse semigroup from an inversive category

Having functorially associated an inversive category with a given inverse semigroup in the previous section, we proceed to show that every inversive category arises as the category $\mathbb{L}(S)$ of a suitable inverse semigroup S . Naturally, we search for our required inverse semigroup in the set of all inversive cones arising from an inversive category. First, we need to prove some preliminary lemmas.

Lemma 3.6. *If an so-category \mathcal{C} is unambiguous, then a cone γ in \mathcal{C} is inversive if and only if it can be represented as $\gamma = \mu * u$, where μ is an idempotent inversive cone and u is an isomorphism in \mathcal{C} .*

Proof. Let γ be an inversive cone. Then $\gamma(m_\gamma)$ is an isomorphism. Let $u = \gamma(m_\gamma)$ and $\mu = \gamma * u^{-1}$. By definition $\gamma = \gamma * (u^{-1}u) = (\gamma * u^{-1}) * u = \mu * u$. Also since $c_\mu = m_\gamma$, we have $\mu(c_\mu) = \gamma * u^{-1}(m_\gamma) = \gamma(m_\gamma)u^{-1} = uu^{-1} = 1_{c_\mu}$. Hence, μ is an idempotent cone.

Now it remains to show that μ is inversive. Suppose μ does not satisfy condition (1) of Definition 3.3. Then there exists an object $d \in v\mathcal{C}$ such that $d \neq m_\gamma$ and $\mu(d)$ is an isomorphism. Then $\mu(d) = \gamma(d)u^{-1}$ is an isomorphism. That is $\gamma(d)$ is

an isomorphism and $d \neq m_\gamma$. This is a contradiction to the fact that γ is inversive and hence, μ satisfies Definition 3.3(1).

Also, for an arbitrary $c \in v\mathcal{C}$, observe that

$$\begin{aligned} \mu(c) &= \gamma(c)u^{-1} \\ &= q(c, c \wedge m_\gamma)\iota(c \wedge m_\gamma, m_\gamma)\gamma(m_\gamma)(\gamma(m_\gamma))^{-1} \quad \text{by Remark 3.3} \\ &= q(c, c \wedge m_\gamma)\iota(c \wedge m_\gamma, m_\gamma). \end{aligned}$$

Hence, using the unique normal factorization property, we have $\text{coim } \mu(c) = c \wedge m_\gamma = c \wedge c_\mu = c \wedge m_\mu$. Thus, μ is an inversive cone.

Conversely, suppose $\gamma = \mu * u$, where μ is an idempotent inversive cone and u is an isomorphism in \mathcal{C} . Then arguing similarly as above, we can see γ is a cone satisfying Definition 3.3(1) with $m_\gamma = c_\mu = m_\mu$ and $\gamma(m_\gamma) = u$. Also, for an arbitrary $c \in v\mathcal{C}$, observe that

$$\begin{aligned} \gamma(c) &= \mu(c)u \\ &= q(c, c \wedge m_\mu)\iota(c \wedge m_\mu, m_\mu)u \quad \text{by Remark 3.3} \\ &= q(c, c \wedge m_\gamma)\iota(c \wedge m_\gamma, m_\gamma)\gamma(m_\mu) \quad \text{since } m_\gamma = m_\mu. \end{aligned}$$

Since $\iota(c \wedge m_\gamma, m_\gamma)\gamma(m_\mu)$ is a monomorphism, using Proposition 2.4, we have $\text{coim } \gamma(c) = c \wedge m_\gamma$. Hence, γ is an inversive cone. □

Remark 3.7. In an inversive category, since there is a unique associated inversive idempotent cone μ_c for each object c , the above lemma can be strengthened as follows: a cone γ is inversive if and only if γ can be uniquely represented as $\gamma = \mu_{m_\gamma} * \gamma(m_\gamma)$.

Lemma 3.8. *Let \mathcal{C} be an inversive category. Given an idempotent inversive cone μ and a retraction $e : c_\mu \rightarrow d$, the cone $\nu = \mu * e$ is an idempotent inversive cone with apex d .*

Proof. Since μ is an idempotent inversive cone, $\mu(c) = q(c, c \wedge c_\mu)\iota(c \wedge c_\mu, c_\mu)$ because $\mu(c_\mu) = 1_{c_\mu}$ (see Remark 3.3). Then

$$\begin{aligned} \nu(d) &= \mu * e(d) = \mu(d)e \\ &= q(d, d \wedge c_\mu)\iota(d \wedge c_\mu, c_\mu)q(c_\mu, d) \quad \text{since } e = q(c_\mu, d) \\ &= q(d, d)\iota(d, c_\mu)q(c_\mu, d) \quad \text{since } d \wedge c_\mu = d \\ &= 1_d \quad \text{since } \iota(d, c_\mu)q(c_\mu, d) = 1_d. \end{aligned}$$

Hence, ν is an idempotent cone with apex d . Now if there exists $d' \in m_\nu$, then $\nu(d')$ is an isomorphism. Then

$$\nu(d') = \mu(d')q = q(d', d' \wedge c_\mu)\iota(d' \wedge c_\mu, c_\mu)q(c_\mu, d).$$

Now since \mathcal{C} is inversive and $\nu(d') \in \langle \mathcal{C} \rangle$, by (IC 4) and Remark 3.1, we have $d' = d$ and $\nu(d') = 1_d$. Hence, ν satisfies Definition 3.3(1).

Also, $\nu(c) = q(c, c \wedge c_\mu) \iota(c \wedge c_\mu, c_\mu) q(c_\mu, c_\nu)$ and so $\nu(c) \in \langle \mathcal{C} \rangle$. By (IC 4), the morphism $\nu(c)$ has a unique inversive factorization. Since $c \wedge c_\mu \wedge c_\nu = c \wedge (c_\mu \wedge c_\nu) = c \wedge c_\nu$, we have

$$\nu(c) = q(c, c \wedge c_\nu) \iota(c \wedge c_\nu, c_\nu).$$

Hence, $\text{coim } \nu(c) = c \wedge c_\nu = c \wedge m_\nu$ and the cone ν satisfies Definition 3.3(2). \square

Now, given an inversive category \mathcal{C} , we build an inverse semigroup from its inversive cones. Let γ, δ be inversive cones in \mathcal{C} . As in (2.5), we define

$$\gamma \cdot \delta := \gamma * (\delta(c_\gamma))^\circ,$$

where $(\delta(c_\gamma))^\circ$ is the epimorphic component of the morphism $\delta(c_\gamma)$. Then clearly $\gamma \cdot \delta$ is a cone. We need to verify that it is an inversive cone.

Proposition 3.9. *Let \mathcal{C} be an inversive category. Then $\gamma \cdot \delta$ as defined above is an inversive cone.*

Proof. By Lemma 3.6, it suffices to show that $\gamma \cdot \delta$ can be represented as $\mu * u$, where μ is an idempotent inversive cone and u is an isomorphism in \mathcal{C} . Since γ is an inversive cone, $\nu := \gamma * (\gamma(m_\gamma))^{-1}$ is an idempotent cone with apex m_γ (as in the proof of Lemma 3.8). Now observe that

$$\gamma \cdot \delta = \gamma * (\delta(c_\gamma))^\circ = \gamma * (\gamma(m_\gamma))^{-1} \gamma(m_\gamma) (\delta(c_\gamma))^\circ = \nu * \gamma(m_\gamma) (\delta(c_\gamma))^\circ.$$

Then since $\gamma(m_\gamma)$ is an isomorphism and $(\delta(c_\gamma))^\circ$ is an epimorphism, the morphism $\gamma(m_\gamma) (\delta(c_\gamma))^\circ$ is an epimorphism and using Proposition 2.4, it has a unique normal factorization of the form qu , where q is a retraction and u is an isomorphism. Let $\mu = \nu * q$. Then

$$\gamma \cdot \delta = \nu * qu = (\nu * q) * u = \mu * u.$$

By Lemma 3.8, the idempotent cone μ is inversive and so by Lemma 3.6, we see that $\gamma \cdot \delta$ is an inversive cone. \square

Thus, given an inversive category \mathcal{C} , the set $\tilde{\mathcal{C}}$ of all its inversive cones forms a subsemigroup of the regular semigroup $\hat{\mathcal{C}}$ of all its normal cones.

Proposition 3.10. *The subsemigroup $\tilde{\mathcal{C}}$ is inverse.*

Proof. Given an inversive cone $\gamma \in \tilde{\mathcal{C}}$ with apex d , Lemma 2.10(1) implies that every idempotent cone μ such that $\mu \mathcal{L} \gamma$ in $\hat{\mathcal{C}}$ satisfies $c_\mu = d$. By (IC 5), there exists a unique idempotent inversive cone with this property. Hence, every \mathcal{L} -class in $\tilde{\mathcal{C}}$ contains a unique idempotent.

Now, we proceed to show that the \mathcal{R} -class of an inversive cone γ in $\tilde{\mathcal{C}}$ contains a unique idempotent. Clearly, $\gamma * (\gamma(m_\gamma))^{-1}$ is an idempotent in the \mathcal{R} -class of the

cone γ (see Remark 3.7). Now, we argue by contradiction. Suppose, there exist two distinct idempotent cones, namely ν and μ in the \mathcal{R} -class of an inversive cone γ . Observe that since an \mathcal{H} -class can contain at most one idempotent, the idempotents ν and μ are not \mathcal{L} -related and hence by Lemma 2.10(1), $c_\nu \neq c_\mu$. Then using Lemma 2.10(2), we can see that $\gamma(c_\nu)$ and $\gamma(c_\mu)$ are both isomorphisms. Hence, the M -set of the inversive cone γ contains two distinct objects c_ν and c_μ . This is a contradiction and hence the \mathcal{R} -class of an inversive cone γ in $\tilde{\mathcal{C}}$ contains a unique idempotent.

By Theorem 2.1, the semigroup $\tilde{\mathcal{C}}$ is inverse. □

The following are immediate specializations of Theorem 2.13 and Corollary 2.14.

Theorem 3.11. *Let \mathcal{C} be an inversive category and $\tilde{\mathcal{C}}$ the inverse semigroup of its inversive cones. Define a functor Ψ between the categories \mathcal{C} and $\mathbb{L}(\tilde{\mathcal{C}})$ as follows:*

$$v\Psi(c) := \tilde{\mathcal{C}}_\mu \quad \text{for } c \in v\mathcal{C},$$

$$\Psi(f) := \rho(\mu, \mu * f^\circ, \nu) \quad \text{for } f \in \mathcal{C}(c, d),$$

where $\mu, \nu \in E(\tilde{\mathcal{C}})$ are such that $c_\mu = c$ and $c_\nu = d$. Then Ψ is an isomorphism of inversive categories.

Corollary 3.12. *A category is inversive if and only if it is isomorphic to the category $\mathbb{L}(S)$ for some inverse semigroup S .*

Remark 3.13. Observe that we can prove the exact dual results for the category $\mathbb{R}(S)$ of principal right ideals of an inverse semigroup S .

Thus, given an inversive category \mathcal{C} , we have an associated inverse semigroup $\tilde{\mathcal{C}}$. Now, we proceed to show that this association is also functorial. To this end, we begin with the following lemma which easily follows from Lemma 3.6.

Lemma 3.14. *Suppose Φ is an inversive functor between two inversive categories \mathcal{C}_1 and \mathcal{C}_2 , if $\gamma = \mu_{m_\gamma} * u$ is an inversive cone in \mathcal{C}_1 , then*

$$\Phi(\gamma) := \mu_{v\Phi(m_\gamma)} * \Phi(u)$$

is an inversive cone in the inversive category \mathcal{C}_2 .

Lemma 3.15. *If Φ is an inversive functor between two inversive categories \mathcal{C}_1 and \mathcal{C}_2 , then the mapping $\phi : \tilde{\mathcal{C}}_1 \rightarrow \tilde{\mathcal{C}}_2$ defined as*

$$\phi : \gamma \mapsto \Phi(\gamma)$$

is a semigroup homomorphism.

Proof. By Remark 3.7 and Lemma 3.14, the mapping ϕ is well defined. Also, for $\gamma_1, \gamma_2 \in \tilde{\mathcal{C}}_1$,

$$\begin{aligned}
 (\gamma_1 \cdot \gamma_2)\phi &= \Phi(\gamma_1 \cdot \gamma_2) \\
 &= \Phi(\gamma_1 * (\gamma_2(c_{\gamma_1}))^\circ) \quad \text{using (2.5)} \\
 &= \Phi(\mu_{m_{\gamma_1}} * u_1(\mu_{m_{\gamma_2}} * u_2(c_{\gamma_1}))^\circ) \quad \text{writing } \gamma_1 = \mu_{m_{\gamma_1}} * u_1 \quad \text{and} \\
 &\qquad\qquad\qquad \gamma_2 = \mu_{m_{\gamma_2}} * u_2 \quad \text{for} \\
 &\qquad\qquad\qquad \text{some isomorphisms } u_1, u_2 \\
 &= \Phi(\mu_{m_{\gamma_1}} * u_1(\mu_{m_{\gamma_2}}(c_{\gamma_1})u_2)^\circ) \quad \text{using (2.4)} \\
 &= \Phi(\mu_{m_{\gamma_1}} * u_1qu) \quad \text{where } qu \text{ is the normal factorization of} \\
 &\qquad\qquad\qquad (\mu_{m_{\gamma_2}}(c_{\gamma_1})u_2)^\circ \\
 &= \Phi(\mu_{m_{\gamma_1}} * q'u') \quad \text{where } q'u' \text{ is the normal factorization of} \\
 &\qquad\qquad\qquad \text{the epimorphism } u_1qu \\
 &= \Phi(\mu_{c'} * u') \quad \text{letting } \mu_{c'} := \mu_{m_{\gamma_1}} * q' \\
 &= \mu_{v\Phi(c')} * \Phi(u') \quad \text{by definition of } \Phi \\
 &= \mu_{v\Phi(m_{\gamma_1})} * q(v\Phi(m_{\gamma_1}), v\Phi(c'))\Phi(u') \quad \text{using Lemma 3.8} \\
 &= \mu_{v\Phi(m_{\gamma_1})} * \Phi(q')\Phi(u') \quad \text{using (IC 2) in } \mathcal{C}_2 \\
 &= \mu_{v\Phi(m_{\gamma_1})} * \Phi(u_1)\Phi(q)\Phi(u) \quad \text{since inversive functors} \\
 &\qquad\qquad\qquad \text{preserve normal factorizations} \\
 &= \mu_{v\Phi(m_{\gamma_1})} * \Phi(u_1)\Phi((\mu_{m_{\gamma_2}}(c_{\gamma_1})u_2)^\circ) \quad \text{---"} \\
 &= \mu_{v\Phi(m_{\gamma_1})} * \Phi(u_1)(\Phi(\mu_{m_{\gamma_2}}(c_{\gamma_1}))\Phi(u_2))^\circ \quad \text{---"} \\
 &= \mu_{v\Phi(m_{\gamma_1})} * \Phi(u_1)(\mu_{v\Phi(m_{\gamma_2})}(v\Phi(c_{\gamma_1}))\Phi(u_2))^\circ \quad \text{using Lemma 3.14} \\
 &= \mu_{v\Phi(m_{\gamma_1})} * \Phi(u_1)(\mu_{v\Phi(m_{\gamma_2})} * \Phi(u_2)(v\Phi(c_{\gamma_1})))^\circ \quad \text{using (2.4)} \\
 &= (\mu_{v\Phi(m_{\gamma_1})} * \Phi(u_1)) \cdot (\mu_{v\Phi(m_{\gamma_2})} * \Phi(u_2)) \quad \text{using (2.5)} \\
 &= \Phi(\gamma_1) \cdot \Phi(\gamma_2) \quad \text{using Lemma 3.14} \\
 &= \gamma_1\phi \cdot \gamma_2\phi.
 \end{aligned}$$

□

Thus, we have the following proposition which can be easily verified.

Proposition 3.16. *Given an inversive category \mathcal{C} and an inversive functor Φ between two inversive categories \mathcal{C}_1 and \mathcal{C}_2 , the following assignment:*

$$\mathcal{C} \mapsto \tilde{\mathcal{C}}; \quad \Phi \mapsto \phi$$

constitutes a functor, say \mathbf{S} , from the category \mathbf{IC} of inversive categories to the category \mathbf{IS} of inverse semigroups.

3.3. A new structure theorem for inverse semigroups

Having characterized the principal left (right) ideals of an inverse semigroup S as an inversive category, we now proceed to establish a new structure theorem for inverse semigroups. First, recall the following folklore property of inverse semigroups.

Proposition 3.17. *Let S be an inverse semigroup. For every pair p, q of distinct elements in S there exists an idempotent $e \in S$ such that $pe \neq qe$ (respectively, $ep \neq eq$). In particular, S is right reductive.*

Proof. Suppose S is inverse and $p, q \in S$ are such that $pe = qe$ for every idempotent $e \in S$. Taking the idempotent $p^{-1}p$ for e , we conclude that $p = pp^{-1}p = qp^{-1}p$ and similarly, $q = pq^{-1}q$. So

$$p = qp^{-1}p = pq^{-1}qp^{-1}p = pp^{-1}pq^{-1}q = pq^{-1}q = q.$$

Hence, the proposition. □

Theorem 3.18. *If S is an inverse semigroup, the map $\tilde{\rho} : S \rightarrow \widetilde{\mathbb{L}(S)}$ given by $a \mapsto \rho^a$ is a semigroup isomorphism.*

Proof. Using the language of Proposition 2.15, Proposition 3.17 implies that when S is an inverse semigroup, the right regular representation $\rho : S \rightarrow S_\rho$ is injective. Hence, by Proposition 2.15, the map $\tilde{\rho} : S \rightarrow \widetilde{\mathbb{L}(S)}$ given by $a \mapsto \rho^a$ is an injective homomorphism. Also, if $\gamma \in \widetilde{\mathbb{L}(S)}$ is any inversive cone, then by Lemma 3.6, $\gamma = \mu * u$ for some idempotent inversive cone μ and an isomorphism u in $\mathbb{L}(S)$. Clearly, by Lemma 3.4, the only idempotent inversive cones in $\mathbb{L}(S)$ are those of the form ρ^e for some $e \in E(S)$. Also, since any isomorphism in $\mathbb{L}(S)$ is of the form $\rho(e, u, f)$ for $e \mathcal{R} u \mathcal{L} f$, we see that

$$\gamma = \mu * u = \rho^e * \rho(e, u, f) = \rho^{eu} = \rho^u.$$

So $\tilde{\rho}$ is surjective and hence the theorem. □

Now, by Proposition 3.5, we have a functor \mathbf{C} from the category \mathbf{IS} of inverse semigroups to the category \mathbf{IC} of inversive categories and by Proposition 3.16, we have a functor \mathbf{S} from the category \mathbf{IC} of inversive categories to the category \mathbf{IS} of inverse semigroups. We proceed to prove that the functors \mathbf{C} and \mathbf{S} constitute an adjunction between the categories \mathbf{IS} and \mathbf{IC} , leading to a category equivalence.

Observe that for a given inverse semigroup S ,

$$\mathbf{CS}(S) = \mathbf{S}(\mathbb{L}(S)) = \widetilde{\mathbb{L}(S)}.$$

So if we define the map $\psi(S) : S \rightarrow \mathbf{CS}(S)$ as $a \mapsto \rho^a$, then by Theorem 3.18, we see that $\psi(S)$ is a semigroup isomorphism. So, any element of $\mathbf{CS}(S)$ can be denoted

by ρ^u for some $u \in S$. Also for $e \mathcal{R} u \mathcal{L} f$, we see that

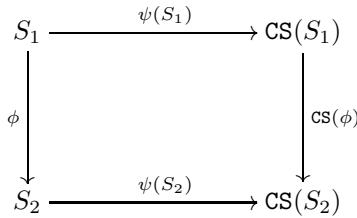
$$\rho^u = \rho^{eu} = \rho^e * \rho(e, u, f).$$

Further, given a homomorphism ϕ between two inverse semigroups S_1 and S_2 , using Proposition 3.5, we see that $\mathbf{C}(\phi) = \Phi$ as defined in (3.1). So, we have a functor $\mathbf{C}(\phi) : \mathbb{L}(S_1) \rightarrow \mathbb{L}(S_2)$. Thus, $\mathbf{S}(\mathbf{C}(\phi))$ is the semigroup homomorphism as defined in Lemma 3.15 between the semigroups $\widetilde{\mathbb{L}(S_1)}$ and $\widetilde{\mathbb{L}(S_2)}$ induced by the functor $\mathbf{C}(\phi)$. Hence, for an element $\rho^u = \rho^e * \rho(e, u, f) = \mu_{S_1 e} * \rho(e, u, f) \in \widetilde{\mathbb{L}(S_1)} = \mathbf{CS}(S_1)$,

$$\begin{aligned} (\rho^u)\mathbf{CS}(\phi) &= (\mu_{S_1 e} * \rho(e, u, f))\mathbf{CS}(\phi) \\ &= \mu_{v\mathbf{C}(\phi)(S_1 e)} * \mathbf{C}(\phi)(\rho(e, u, f)) \\ &= \mu_{v\Phi(S_1 e)} * \Phi(\rho(e, u, f)) \\ &= \mu_{S_2 e\phi} * \rho(e\phi, u\phi, f\phi) \\ &= \rho^{e\phi} * \rho(e\phi, u\phi, f\phi) \\ &= \rho^{e\phi u\phi} = \rho^{eu\phi} = \rho^{u\phi}. \end{aligned}$$

Proposition 3.19. *Given an inverse semigroup S , the map $\psi : S \mapsto \psi(S)$ is a natural isomorphism from the identity functor $1_{\mathbf{IS}}$ to the functor \mathbf{CS} .*

Proof. Since $\psi(S)$ is a semigroup isomorphism, it suffices to show that ψ is a natural transformation. That is, for a homomorphism $\phi : S_1 \rightarrow S_2$, we need to show that the following diagram commutes.



For $a \in S_1$, we have

$$a\phi\psi(S_2) = \rho^{a\phi}.$$

Also, from the discussion above, we have

$$a\psi(S_1)\mathbf{CS}(\phi) = (\rho^a)\mathbf{CS}(\phi) = \rho^{a\phi}.$$

So, the above diagram commutes and hence ψ is a natural isomorphism. □

Finally, we need to show that the identity functor $1_{\mathbf{IC}}$ is naturally isomorphic to the functor \mathbf{SC} . Observe that for a given inversive category \mathcal{C} ,

$$\mathbf{SC}(\mathcal{C}) = \mathbf{C}(\widetilde{\mathcal{C}}) = \mathbb{L}(\widetilde{\mathcal{C}}).$$

So if we define a functor $\Psi(\mathcal{C}) : \mathcal{C} \rightarrow \mathbb{L}(\widetilde{\mathcal{C}})$ as in Theorem 3.11, recall that $\Psi(\mathcal{C})$ is an isomorphism of inversive categories. Given an inversive functor Φ between two

inversive categories \mathcal{C}_1 and \mathcal{C}_2 , then $\mathbf{S}(\Phi)$ as defined in Lemma 3.15 is a semigroup homomorphism from $\tilde{\mathcal{C}}_1$ to $\tilde{\mathcal{C}}_2$. Observe that an arbitrary object of the category $\mathbf{SC}(\mathcal{C}_1) = \mathbb{L}(\tilde{\mathcal{C}}_1)$ may be denoted by $\tilde{\mathcal{C}}_1\vartheta$, where ϑ is an idempotent inversive cone in $\mathbb{L}(\tilde{\mathcal{C}}_1)$. Hence, the functor $\mathbf{SC}(\Phi)$ will map an object $\tilde{\mathcal{C}}_1\vartheta \mapsto \tilde{\mathcal{C}}_2\Phi(\vartheta)$, where $\Phi(\vartheta)$ is defined as in Lemma 3.14.

If $\rho(\vartheta, \xi, \nu)$ is an arbitrary morphism of the category $\mathbb{L}(\tilde{\mathcal{C}}_1)$, where ϑ, ν are idempotent inversive cones in $\mathbb{L}(\tilde{\mathcal{C}}_1)$ and $\xi \in \vartheta\tilde{\mathcal{C}}_1\nu$, the functor $\mathbf{SC}(\Phi)$ map is as follows:

$$\rho(\vartheta, \xi, \nu) \mapsto \rho(\Phi(\vartheta), \Phi(\xi), \Phi(\nu)),$$

where $\Phi(\vartheta)$ etc. are defined as in Lemma 3.14.

Proposition 3.20. *Given an inversive category \mathcal{C} , the map $\Psi : \mathcal{C} \mapsto \Psi(\mathcal{C})$ is a natural isomorphism from the identity functor $1_{\mathbf{IC}}$ to the functor \mathbf{SC} .*

Proof. Since $\Psi(\mathcal{C})$ is an isomorphism, it suffices to show that Ψ is a natural transformation. That is, for a functor $\Phi : \mathcal{C}_1 \rightarrow \mathcal{C}_2$, we need to show that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{\Psi(\mathcal{C}_1)} & \mathbf{SC}(\mathcal{C}_1) \\ \Phi \downarrow & & \downarrow \mathbf{sc}(\Phi) \\ \mathcal{C}_2 & \xrightarrow{\Psi(\mathcal{C}_2)} & \mathbf{SC}(\mathcal{C}_2) \end{array}$$

For an object $c \in \nu\mathcal{C}_1$, we have

$$\begin{aligned} \Phi\Psi(\mathcal{C}_2)(c) &= \Psi(\mathcal{C}_2)(\Phi(c)) \\ &= \tilde{\mathcal{C}}_2\mu \quad (\text{where } \mu \text{ is an idempotent inversive cone such that } c_\mu = \Phi(c)). \end{aligned}$$

Also, from the discussion above, we have

$$\begin{aligned} \Psi(\mathcal{C}_1)\mathbf{SC}(\Phi)(c) &= \mathbf{SC}(\Phi)(\tilde{\mathcal{C}}_1\vartheta) \quad (\text{where } c_\vartheta = c) \\ &= \tilde{\mathcal{C}}_2\Phi(\vartheta) \quad (\text{where } \Phi(\vartheta) \text{ is an inversive cone as in Lemma 3.14}). \end{aligned}$$

By the definition of $\Phi(\vartheta)$ as in Lemma 3.14, it is clear that if ϑ is an idempotent inversive cone in \mathcal{C}_1 with apex c , then $\Phi(\vartheta)$ is an idempotent inversive cone \mathcal{C}_2 with apex $\Phi(c)$. Moreover, since there is a unique idempotent inversive cone with a given vertex, we conclude that $\mu = \Phi(\vartheta)$. Hence,

$$\Phi\Psi(\mathcal{C}_2)(c) = \Psi(\mathcal{C}_1)\mathbf{SC}(\Phi)(c)$$

and so the diagram commutes for every object $c \in \nu\mathcal{C}_1$.

Further for a morphism $f : c \rightarrow d$ in the category \mathcal{C}_1 , we have

$$\Phi\Psi(\mathcal{C}_2)(f) = \Psi(\mathcal{C}_2)(\Phi(f)) = \rho(\mu, \mu * (\Phi(f))^\circ, \nu),$$

where μ and ν are idempotent inversive cones such that $c_\mu = \Phi(c)$ and $c_\nu = \Phi(d)$. Also,

$$\begin{aligned} \Psi(\mathcal{C}_1)\text{SC}(\Phi)(f) &= \text{SC}(\Phi)(\rho(\vartheta, \vartheta * f^\circ, \upsilon)) \quad \text{where } \vartheta, \upsilon \text{ are idempotent inversive} \\ &\quad \text{cones in } \mathbb{L}(\tilde{\mathcal{C}}_1) \text{ such that } c_\vartheta=c \text{ and } c_\upsilon=d \\ &= \rho(\Phi(\vartheta), \Phi(\vartheta * f^\circ), \Phi(\upsilon)) \quad \text{where } \Phi(\vartheta) \text{ etc. are as in} \end{aligned}$$

Lemma 3.14.

As argued in the case of objects, we can easily see that $\mu = \Phi(\vartheta)$ and $\nu = \Phi(\upsilon)$. Further, since Φ is an inversive functor, as argued in the proof of Lemma 3.15, we can verify that $\mu * (\Phi(f))^\circ = \Phi(\vartheta * f^\circ)$. That is, for a morphism $f : c \rightarrow d$ in the category \mathcal{C}_1 ,

$$\Phi\Psi(\mathcal{C}_2)(f) = \Psi(\mathcal{C}_1)\text{SC}(\Phi)(f)$$

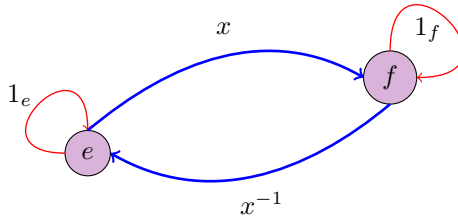
and the diagram commutes for every morphism f in \mathcal{C}_1 . So, the diagram is commutative and hence Ψ is a natural isomorphism. □

Combining Propositions 3.19 and 3.20, we arrive at the main result of this section.

Theorem 3.21. *The category **IS** of inverse semigroups is equivalent to the category **IC** of inversive categories.*

3.4. Inductive groupoids and inversive categories

Any element x in an inverse semigroup S can be seen as a *morphism* from the idempotent $e := xx^{-1}$ to the idempotent $f := x^{-1}x$.



In this manner, one can naturally associate a groupoid $\mathcal{G}(S)$ with a given inverse semigroup S such that the set $v\mathcal{G}(S)$ of objects coincides with the set of idempotents of the semigroup S . Abstracting the characteristic properties of the groupoid $\mathcal{G}(S)$ leads to the following definition.

Definition 3.5. Let \mathcal{G} be a groupoid and denote by $\mathbf{d} : \mathcal{G} \rightarrow v\mathcal{G}$ and $\mathbf{r} : \mathcal{G} \rightarrow v\mathcal{G}$ its domain and codomain maps, respectively. Let \leq be a partial order on \mathcal{G} . Then (\mathcal{G}, \leq) is called an *inductive groupoid* if $(v\mathcal{G}, \leq)$ is a semilattice and for all $e, f \in v\mathcal{G}$ and all $x, y, u, v \in \mathcal{G}$, the following hold:

- (IG 1) If $u \leq x, v \leq y$ and $\mathbf{r}(u) = \mathbf{d}(v), \mathbf{r}(x) = \mathbf{d}(y)$, then $uv \leq xy$.
- (IG 2) If $x \leq y$, then $x^{-1} \leq y^{-1}$.
- (IG 3) If $1_e \leq 1_{\mathbf{d}(x)}$, then there exists a unique morphism $e \downarrow x \in \mathcal{G}$ (called the *restriction of x to e*) such that $e \downarrow x \leq x$ and $\mathbf{d}(e \downarrow x) = e$.
- (IG 3*) If $1_f \leq 1_{\mathbf{r}(x)}$, then there exists a unique morphism $x \downarrow f \in \mathcal{G}$ (called the *corestriction of x to f*) such that $x \downarrow f \leq x$ and $\mathbf{r}(x \downarrow f) = f$.

The inductive groupoids with inductive functors as morphisms form the locally small category **IG** of inductive groupoids, see [17, Chap. 4] for details. The above discussed association between semigroups and groupoids can be extended to a category isomorphism (not just a category equivalence) as follows.

Theorem 3.22 (Ehresmann–Schein–Nambooripad Theorem, see [17, Theorem 4.1.8]). *The category **IG** of inductive groupoids is isomorphic to the category **IS** of inverse semigroups.*

Using Theorems 3.22 and 3.21, by transitivity, we see that the category **IG** of inductive groupoids is equivalent to the category **IC** of inversive categories. Now, we proceed to describe a direct category equivalence between the categories **IG** and **IC**, without any semigroup assumptions. This may be seen as a very much simplified version of the results in [9, 10].

First, given an inversive category \mathcal{C} with the semilattice order \leq , we proceed to identify the inductive groupoid associated with \mathcal{C} . To this end, let $\mathcal{G}_{\mathcal{C}}$ be the subcategory of the category \mathcal{C} consisting of all isomorphisms in \mathcal{C} . Clearly, $\mathcal{G}_{\mathcal{C}}$ is a groupoid. Given any two morphisms φ and ψ in $\mathcal{G}_{\mathcal{C}}$ with domains c and d , respectively, define a relation $\leq_{\mathcal{C}}$ as follows:

$$\varphi \leq_{\mathcal{C}} \psi \Leftrightarrow c \leq d \quad \text{and} \quad \varphi = (\iota(c, d)\psi)^{\circ},$$

where $\iota(c, d)$ is the inclusion from c to d and $(\iota(c, d)\psi)^{\circ}$ is the epimorphic component of the monomorphism $\iota(c, d)\psi$ in the inversive category \mathcal{C} . It can be easily seen that $\leq_{\mathcal{C}}$ is a partial order on $\mathcal{G}_{\mathcal{C}}$.

Since the order $\leq_{\mathcal{C}}$ reduces to the semilattice order \leq on the identities of $\mathcal{G}_{\mathcal{C}}$, we observe that $(v\mathcal{G}_{\mathcal{C}}, \leq_{\mathcal{C}})$ forms a semilattice. Further, given a morphism ψ in $\mathcal{G}_{\mathcal{C}}$ with domain d and if $1_c \leq_{\mathcal{C}} 1_d$, then by letting

$$c \downarrow \psi := (\iota(c, d)\psi)^{\circ}$$

as the restriction of the morphism ψ in the groupoid $\mathcal{G}_{\mathcal{C}}$ to the object c , we can easily verify the following.

Proposition 3.23. *$(\mathcal{G}_{\mathcal{C}}, \leq_{\mathcal{C}})$ is an inductive groupoid.*

Also, given an inversive functor between two inversive categories, its restriction will give an inductive functor between the two corresponding inductive groupoids. Thus, the above correspondence is functorial between the categories **IC** and **IG**.

Conversely, given an inductive groupoid (\mathcal{G}, \leq) , we proceed to ‘build’ the inversive category $\mathcal{C}_{\mathcal{G}}$ associated with it. To this end, we will follow the scheme used in [10, Sec. 4]. We consider three intermediary categories: $\mathcal{P}_{\mathcal{G}}$, \mathcal{G} and $\mathcal{Q}_{\mathcal{G}}$ responsible for inclusions, isomorphisms and retractions, respectively. Then we introduce a special partial binary operation to build the required category $\mathcal{C}_{\mathcal{G}}$ from these categories. A somewhat similar construction can be seen in [28, Theorem 3.4].

Given an inductive groupoid (\mathcal{G}, \leq) , the categories $\mathcal{P}_{\mathcal{G}}$ and $\mathcal{Q}_{\mathcal{G}}$ are such that

$$v\mathcal{P}_{\mathcal{G}} = v\mathcal{Q}_{\mathcal{G}} := v\mathcal{G}.$$

Also, for $e, f \in v\mathcal{G}$ such that $e \leq f$, a morphism in $\mathcal{P}_{\mathcal{G}}$ is defined as the unique morphism $\iota(e, f)$ from e to f and a morphism in $\mathcal{Q}_{\mathcal{G}}$ is defined as the unique morphism $q(f, e)$ from f to e . The category $\mathcal{P}_{\mathcal{G}}$ is a strict preorder and its morphisms are called inclusions.

In the case of general normal categories in [9, 10], a further auxiliary category was needed. Here, we do not need it since inversive categories have unique factorizations and this allows us to use \mathcal{G} itself as a building block. So, the set of objects of the required category is $v\mathcal{C}_{\mathcal{G}} := v\mathcal{G}$ and morphisms in $\mathcal{C}_{\mathcal{G}}$ are defined by

$$\mathcal{C}_{\mathcal{G}} := \{(q, \alpha, j) \in \mathcal{Q}_{\mathcal{G}} \times \mathcal{G} \times \mathcal{P}_{\mathcal{G}} : \mathbf{r}(q) = \mathbf{d}(\alpha) \text{ and } \mathbf{r}(\alpha) = \mathbf{d}(j)\}.$$

As in [10, Sec. 4], we denote an arbitrary morphism (q, α, j) in $\mathcal{C}_{\mathcal{G}}$ by just $[e, \alpha, f]$, where $e = \mathbf{d}(q)$ and $f = \mathbf{r}(j)$. Given two such morphisms $[e, \alpha, f], [f, \beta, g] \in \mathcal{C}_{\mathcal{G}}$, we compose them as follows. For $h = \mathbf{r}(\alpha) \wedge \mathbf{d}(\beta)$,

$$[e, \alpha, f][f, \beta, g] = [e, \alpha|h \cdot h|\beta, g]. \tag{3.2}$$

Then $\mathcal{C}_{\mathcal{G}}$ forms a category such that $\mathcal{P}_{\mathcal{G}}$ is a strict preorder subcategory by identifying any morphism $j \in \mathcal{P}_{\mathcal{G}}$ with $[\mathbf{d}(j), 1_{\mathbf{d}(j)}, \mathbf{r}(j)] \in \mathcal{C}_{\mathcal{G}}$. Similarly, $\mathcal{Q}_{\mathcal{G}}$ is a subcategory of $\mathcal{C}_{\mathcal{G}}$ by identifying any morphism $q \in \mathcal{Q}_{\mathcal{G}}$ with $[\mathbf{d}(q), 1_{\mathbf{r}(q)}, \mathbf{r}(q)] \in \mathcal{C}_{\mathcal{G}}$. It is easy to verify that $(\mathcal{C}_{\mathcal{G}}, \mathcal{P}_{\mathcal{G}})$ satisfies (IC 1)–(IC 3).

Let $f = \iota(e_1, e_2)q(e_2, e_3) \cdots \iota(e_{2n-1}, e_{2n})q(e_{2n}, e_{2n+1})$ be an arbitrary morphism in the core $\langle \mathcal{C}_{\mathcal{G}} \rangle$ so that

$$\begin{aligned} f &= [e_1, 1_{e_1}, e_2][e_2, 1_{e_3}, e_3][e_3, 1_{e_3}, e_4] \cdots [e_{2n-1}, 1_{e_{2n-1}}, e_{2n-1}][e_{2n}, 1_{e_{2n+1}}, e_{2n+1}] \\ &= [e_1, 1_{e_1 \wedge e_3}, e_3][e_3, 1_{e_3}, e_4] \cdots [e_{2n-1}, 1_{e_{2n-1}}, e_{2n-1}][e_{2n}, 1_{e_{2n+1}}, e_{2n+1}] \\ &= [e_1, 1_{e_1 \wedge e_3}, e_4] \cdots [e_{2n-1}, 1_{e_{2n-1}}, e_{2n-1}][e_{2n}, 1_{e_{2n+1}}, e_{2n+1}] \\ &= [e_1, 1_e, e_{2n+1}] \quad \text{where } e = \bigwedge_{i=0}^n e_{2i+1} \\ &= [e_1, 1_e, e][e, 1_e, e_{2n+1}]. \end{aligned}$$

Hence, any morphism in $\langle \mathcal{C}_{\mathcal{G}} \rangle$ admits an inversive factorization and so (IC 4) is satisfied.

Finally, given $\alpha \in \mathcal{G}$, if we define a map $r^\alpha : v\mathcal{C}_\mathcal{G} \rightarrow \mathcal{C}_\mathcal{G}$ as

$$r^\alpha : g \mapsto [g, h | \alpha, \mathbf{r}(\alpha)],$$

where $h = g \wedge \mathbf{d}(\alpha)$, we can easily verify that r^α is an inversive cone with apex $\mathbf{r}(\alpha)$ and $m_{r^\alpha} = \mathbf{d}(\alpha)$. Hence, for any object $e \in v\mathcal{C}_\mathcal{G}$, if we define r^e as

$$r^e : g \mapsto [g, 1_h, e],$$

where $h = g \wedge e$, then r^e is the unique idempotent inversive cone with apex e . Thus (IC 5) is also satisfied. Hence, we have the following proposition.

Proposition 3.24. *$(\mathcal{C}_\mathcal{G}, \mathcal{P}_\mathcal{G})$ is an inversive category.*

As in [10, Sec. 4], we can easily show that the discussed correspondence between the categories **IG** and **IC** is also functorial. This leads to the following theorem.

Theorem 3.25. *The category **IG** of inductive groupoids is equivalent to the category **IC** of inversive categories.*

Once the equivalence of the categories **IG** and **IC** has been established, the equivalence between the categories of inverse semigroups and inductive groupoids becomes a consequence of Theorem 3.21. Thus, we have recovered a weak version of the Ehresmann–Schein–Nambooripad Theorem from our consideration. (Recall that the ‘full’ Ehresmann–Schein–Nambooripad Theorem claims that the two latter categories are isomorphic rather than equivalent).

4. Completely 0-Simple Semigroups

In this section, we take a slightly different perspective and discuss how the abstract construction described in Sec. 2 simplifies drastically in the case of completely 0-simple semigroups. This section may also be seen as a relatively straightforward generalization of the discussion in [7] wherein the cross-connection structure of completely simple semigroups was studied in great detail. So whenever an exact repetition of the argument suffices, without further comments, we refer the reader to [7] for the details of the results outlined here. The ensuing discussion having a relatively low entry threshold may also act as a gateway to cross-connection theory for beginners.

It is known [32] that a completely 0-simple semigroup is isomorphic to the Rees matrix semigroup $\mathcal{M}^\circ(G; I, L; P)$ described as follows. We set $\mathcal{M}^\circ(G; I, L; P) = (I \times G \times L) \cup \{0\}$, where G is a group, I and L are sets and $P = (p_{\ell i})$ is an $L \times I$ matrix with entries in $G^\circ := G \cup \{0\}$ such that no row or column of P consists entirely of zeros (then P is called a *regular sandwich matrix*). The multiplication

in $\mathcal{M}^\circ(G; I, L; P)$ is given by

$$(i, a, \ell)(j, b, k) := \begin{cases} (i, ap_{\ell_j}b, k) & \text{if } p_{\ell_j} \neq 0, \\ 0 & \text{if } p_{\ell_j} = 0, \end{cases} \tag{4.1}$$

$$(i, a, \ell)0 = 0(i, a, \ell) = 00 := 0.$$

Now, we proceed to specialize the abstract results of Sec. 2. To that end, we fix a Rees matrix semigroup $\mathcal{M}^\circ(G; I, L; P)$ and denote it by just S . So, in this case, as the reader shall see, the categories and the various abstract objects in the cross-connection description of a semigroup get concrete descriptions in terms of the sets I and L , group G and the sandwich matrix P . So, we proceed to define two concrete categories \mathfrak{L} and \mathfrak{R} , and show that they are cross-connected via the sandwich matrix P . Finally, we use this discussion to formulate a structure theorem.

First, since the principal left ideals of a Rees matrix semigroup $\mathcal{M}^\circ(G; I, L; P)$ semigroup are characterized using the set L , the object set of the category \mathfrak{L} of principal left ideals may be identified as follows:

$$v\mathfrak{L} := L^\circ = L \cup \{0\}. \tag{4.2}$$

In the sequel, any arbitrary principal left ideal $S(i, a, \ell)$ shall be represented by just ℓ whenever there is no confusion. As with general normal categories, here also the object set shall be treated as a strict preorder subcategory as described below. Given $\ell_1, \ell_2 \in v\mathfrak{L}$ such that $\ell_1, \ell_2 \neq 0$ and for each $g \in G$, we have a morphism from ℓ_1 to ℓ_2 denoted by $\rho(\ell_1, g, \ell_2)$ such that it maps 0 to 0 and for each $(i, a, \ell_1) \in S(i_1, a_1, \ell_1)$,

$$\rho(\ell_1, g, \ell_2) : (i, a, \ell_1) \mapsto (i, ag, \ell_2) \in S(i_2, a_2, \ell_2).$$

The abuse of notation must be noted here that since the morphism set is in bijection with the set ‘ eSf ’, we are in fact ‘absorbing’ the sandwich element into translating element and denoting the morphism using the single group element g . Also, $\rho(\ell_1, 0, \ell_2)$ is a morphism from ℓ_1 to ℓ_2 defined by

$$\rho(\ell_1, 0, \ell_2) : (i, a, \ell_1) \mapsto 0 \in S(i_2, a_2, \ell_2).$$

Thus, $\mathfrak{L}(\ell_1, \ell_2) := G^\circ$ if $\ell_1, \ell_2 \neq 0$. Observe that L° is a strict preorder and the only nontrivial inclusions arise from the relation $0 \subseteq \ell$. So, for each $\ell \neq 0$ in the set \mathfrak{L} , we have an inclusion $\rho(0, 0, \ell)$ from 0 to ℓ . Further, for each inclusion $\rho(0, 0, \ell) \in \mathfrak{L}(0, \ell)$, we have a *unique* retraction $\rho(\ell, 0, 0) \in \mathfrak{L}(\ell, 0)$ and thus every inclusion in \mathfrak{L} splits uniquely. Finally, the only morphism in $\mathfrak{L}(0, 0)$ may be denoted by $\rho(0, 0, 0)$. Thus, the composition of the morphisms in \mathfrak{L} is as described in the earlier sections:

$$\rho(\ell_1, g, \ell_2)\rho(\ell_2, h, \ell_3) = \rho(\ell_1, gh, \ell_3),$$

where $\ell_1, \ell_2, \ell_3 \in L^\circ$ and $g, h \in G^\circ$. In the sequel, whenever there is no ambiguity regarding the domain and codomain of the morphism, we represent an arbitrary morphism $\rho(\ell_1, g, \ell_2)$ in \mathfrak{L} by just ρ_g . It can be easily verified that \mathfrak{L} as described above forms a category with subobjects.

Observe that an arbitrary morphism $\rho(\ell_1, g, \ell_2) \in \mathfrak{L}$ either is an isomorphism wherein $\ell_1, \ell_2, g \neq 0$ or has a unique normal factorization of the form

$$\rho(\ell_1, 0, \ell_2) = \rho(\ell_1, 0, 0)\rho(0, 0, \ell_2).$$

Hence, every morphism in \mathfrak{L} has a unique normal factorization.

Now, we proceed to describe the normal cones in \mathfrak{L} . As usual, given sets A and B , the set of all functions $f : B \rightarrow A$ is denoted A^B . It is handy to represent $f \in A^B$ as a $|B|$ -tuple of elements of A of the form $(a_\alpha)_{\alpha \in B}$, where $a_\alpha := f(\alpha)$.

The set $T := (G^\circ)^L \times L$ forms a semigroup under the following multiplication. Given $\gamma = ((g_\alpha)_{\alpha \in L}; \ell)$, $\delta = ((h_\alpha)_{\alpha \in L}; k) \in T$ such that $(g_\alpha)_{\alpha \in L}, (h_\alpha)_{\alpha \in L} \in (G^\circ)^L$ and $\ell, k \in L$,

$$\gamma * \delta := ((g_\alpha h_\ell)_{\alpha \in L}; k), \tag{4.3}$$

where $h_\ell \in G^\circ$ and $g_\alpha h_\ell$ is the product in G° .

It is clear that normal cones in \mathfrak{L} can be represented as unique elements in T : any normal cone γ with apex $\ell \neq 0$ can be represented by the element $((g_\alpha)_{\alpha \in L}; \ell)$ in T such that for an arbitrary $k \in \mathfrak{L}$,

$$\gamma(k) = \rho(k, g_k, \ell) \quad \text{if } k \neq 0 \quad \text{and} \quad \gamma(0) = \rho(0, 0, \ell).$$

The set $U = \{0\}^L \times L$ is an ideal of T . It consists of elements of T of the form $((0_\alpha)_{\alpha \in L}; \ell)$ which are not normal cones since none of their components are isomorphisms. Taking the Rees quotient $R := T/U$, we can verify that the semigroup $\widehat{\mathfrak{L}}$ of normal cones in \mathfrak{L} is isomorphic to R whereas the zero of R corresponds to the unique normal cone γ_0 in \mathfrak{L} with apex 0, namely $\gamma_0(\ell) = \rho(\ell, 0, 0)$.

We mention in passing that the semigroup T is nothing but the *wreath product* $G^\circ \wr L$ of the 0-group G° and the right zero semigroup L . (Thus, the semigroup $\widehat{\mathfrak{L}}$ is isomorphic to the Rees quotient $(G^\circ \wr L)/(\{0\} \wr L)$). This observation reflects, in a nutshell, the fact that Nambooripad’s somewhat mysterious composition of normal cones (2.5) is actually a sort of wreath product multiplication extended to a general category setting.

In the sequel, by abuse of notation, the image of an element $\gamma \in T$ in the quotient semigroup R will also be denoted by just γ , whenever there is no confusion. So, for an arbitrary element $x = (i, a, \ell)$ in S , the principal cone ρ^x in $\widehat{\mathfrak{L}}$ may be denoted by $((p_{\alpha i} a)_{\alpha \in L}; \ell) \in (G^\circ)^L \times L$. This normal cone ρ^x with apex ℓ is obtained by the right translation of the i th column of P with the nonzero group element a . Since the sandwich matrix P is *regular*, we have $\rho^x \in R$.

Hence, \mathfrak{L} is an unambiguous category which is constructed from a set L and a group G . We shall refer to the categories constructed using such a recipe as *completely unambiguous categories*.

Now, we proceed to characterize the Green relation \mathcal{R} in the semigroup $\widehat{\mathfrak{L}}$; this in turn will provide the description of the unambiguous dual \mathfrak{L}^* . Extending the discussion in [7], we can see that given an arbitrary element $\gamma = ((g_\alpha)_{\alpha \in L}; \ell) \in \widehat{\mathfrak{L}}$,

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the principal right ideal $\gamma\widehat{\mathfrak{L}}$ generated by γ is determined by the $|L|$ -tuple $g_\gamma = (g_\alpha)_{\alpha \in L} \in (G^\circ)^L$ and

$$\gamma\widehat{\mathfrak{L}} = g_\gamma G^\circ \times L.$$

Thus, the \mathcal{R} -classes in the semigroup $\widehat{\mathfrak{L}}$ are in one-one correspondence with the set $\{g_\gamma G^\circ : g_\gamma \in (G^\circ)^L\}$. Observe that the element $(0_\alpha)_{\alpha \in L} G^\circ$ corresponds to the \mathcal{R} -class generated by the zero element γ_0 .

Thus, the set of objects (i.e. the set of H -functors $\{H(\gamma; -) : \gamma \in \widehat{\mathfrak{L}}\}$) of the normal dual \mathfrak{L}^* may be characterized as the set $\{g_\gamma G^\circ : g_\gamma \in (G^\circ)^L\}$. Further, using the fact that the H -functors are representable, we can show that the set of morphisms in \mathfrak{L}^* are in a ‘dual’ correspondence with the set of morphisms in \mathfrak{L} . So the morphisms between nonzero objects in \mathfrak{L}^* can be characterized as the set $(G^\circ)^{\text{op}}$ and in the sequel we denote such an arbitrary morphism by σ_g , where $g \in G^\circ$. Similarly, we can describe the morphisms with the zero object $H(\gamma_0; -)$ as σ_0 .

Having described the categories \mathfrak{L} and \mathfrak{L}^* , using the left-right duality, we can easily describe the completely unambiguous category \mathfrak{R} and its dual \mathfrak{R}^* as follows:

$$v\mathfrak{R} := I^\circ = I \cup \{0\}. \tag{4.4}$$

Any arbitrary principal right ideal $(i, a, \ell)S$ is represented by just i and the set of morphisms $\mathfrak{R}(i_1, i_2) := (G^\circ)^{\text{op}}$ if $i_1, i_2 \neq 0$ wherein an arbitrary morphism is denoted by just λ_g , where $g \in G^\circ$. If $0 \subseteq i$, we have the associated inclusion morphism $\lambda(0, 0, i)$ and the retraction $\lambda(0, 0, i)$; the unique morphism in $\mathfrak{R}(0, 0)$ is $\lambda(0, 0, 0)$.

Then it can be easily shown that the semigroup of normal cones in \mathfrak{R} is given by the Rees quotient $((G^\circ)^I \times I)^{\text{op}} / (\{0\}^I \times I)^{\text{op}}$. Then the object set of the unambiguous dual \mathfrak{R}^* is characterized as $\{G^\circ g_\gamma : g_\gamma \in (G^\circ)^I\}$ and an arbitrary morphism between any two nonzero objects is denoted by τ_g , where $g \in G^\circ$.

Having constructed the completely unambiguous categories \mathfrak{L} and \mathfrak{R} associated with a completely simple semigroup S using the sets L, I and the 0-group G° , now, we proceed to characterize the cross-connection involved.

Observe that given the $L \times I$ matrix P with entries from G° , we can define functors $\Gamma : \mathfrak{R} \rightarrow \mathfrak{L}^*$ and $\Delta : \mathfrak{L} \rightarrow \mathfrak{R}^*$ such that

$$v\Gamma_P : i \mapsto p_i G^\circ, \quad v\Gamma_P : 0 \mapsto \gamma_0 G^\circ, \quad \Gamma_P : \lambda_g \mapsto \sigma_g \quad \text{and} \quad \Gamma_P : \lambda_0 \mapsto \sigma_0; \tag{4.5}$$

$$v\Delta_P : \ell \mapsto G^\circ p_\ell, \quad v\Delta_P : 0 \mapsto G^\circ \delta_0, \quad \Delta_P : \rho_g \mapsto \tau_g \quad \text{and} \quad \Delta_P : \rho_0 \mapsto \tau_0, \tag{4.6}$$

where p_i is the i th column of the matrix P and p_ℓ is the ℓ th row of P . It can be readily seen that these functors constitute a cross-connection. In other words, the sandwich matrix P of the semigroup $\mathcal{M}^\circ(G; I, L; P)$ completely determines the cross-connection functors. The discussion in this section can be summarized in the following theorem.

Theorem 4.1. *Every completely 0-simple semigroup $S = \mathcal{M}^\circ(G; I, L; P)$ determines a cross-connection $(\mathfrak{L}, \mathfrak{R}; \Gamma_P, \Delta_P)$ between the completely unambiguous categories \mathfrak{L} and \mathfrak{R} as described above. Conversely, given two arbitrary sets L, I and a 0-group G° , we can define two completely unambiguous categories \mathfrak{L} and \mathfrak{R} such that an arbitrary regular $L \times I$ matrix P with entries from G° will determine a unique cross-connection $(\mathfrak{L}, \mathfrak{R}; \Gamma_P, \Delta_P)$; the cross-connection semigroup so obtained is the completely 0-simple semigroup $\mathcal{M}^\circ(G; I, L; P)$.*

The correspondence described above can be extended to a category equivalence between the category of completely 0-simple semigroups and the category of cross-connections of completely unambiguous categories.

5. Conclusion and Future Work

We have applied Nambooripad's theory of cross-connections to locally inverse semigroups, obtaining a category equivalence between the category of cross-connected unambiguous categories and the category of locally inverse semigroups. Being specialized to inverse semigroups, our main result leads to a new structure theorem that, as we have shown, is equivalent to a weak version of the Ehresmann–Schein–Nambooripad Theorem; if specialized to completely 0-simple semigroups, the result turns out to constitute a category-theoretic formulation of the Rees Theorem. Thus, we see that the cross-connection approach reveals a common background for the two classical and seemingly unrelated structure theorems.

This observation suggests a promising direction for further investigations aimed to find cross-connections explanations for several known (and possibly yet unknown) structure results for the class of locally inverse semigroups and its important subclasses. In particular, we mean the covering theorems by McAlister [20] and Pastijn–Oliveira [27] that are formulated in terms of the Rees matrix construction.

Considerable attention has been paid in the literature to the structure of the bifree locally inverse semigroup on a set and several related constructs [1–4, 24, 25]. Our result implies that these constructs possess categorical counterparts, which, in our opinion, are worth being studied: this might shed new light on the structure of bifree locally inverse semigroups and their relatives and reveal categorical creatures that might be of independent interest.

A rather unexpected application of the Ehresmann–Schein–Nambooripad Theorem has been recently demonstrated by Malandro [19] who used it for an efficient enumeration of finite inverse semigroups. We wonder if our main result can be applied in a similar fashion for an enumeration of finite locally inverse semigroups.

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