Theoretical Computer Science 922 (2022) 158–169

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

On minimal critical exponent of balanced sequences

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ARTICLE INFO

Article history:
Received 4 December 2021
Received in revised form 15 March 2022
Accepted 21 April 2022
Available online 26 April 2022
Communicated by M. Sciortino

Keywords:
Balanced sequence
Critical exponent
Repetition threshold
Constant gap sequence
Return word
Bispecial factor
Sturmian sequence

ABSTRACT

We study the threshold between avoidable and unavoidable repetitions in infinite balanced sequences over finite alphabets. The conjecture stated by Rampersad, Shallit and Vandomme says that the minimal critical exponent of balanced sequences over the alphabet of size $d \geq 5$ equals $\frac{d-2}{d-3}$. This conjecture is known to hold for $d \in \{5, 6, 7, 8, 9, 10\}$. We refute this conjecture by showing that the picture is different for bigger alphabets. We prove that critical exponents of balanced sequences over an alphabet of size $d \geq 11$ are lower bounded by $\frac{d-1}{d-2}$ and this bound is attained for all even numbers $d \geq 12$. According to this result, we conjecture that the least critical exponent of a balanced sequence over d letters is $\frac{d-1}{d-2}$ for all $d \geq 11$.

1. Introduction

The birth of combinatorics on words is linked to the study of repetitions of factors in infinite words (or sequences) by a Norwegian mathematician Axel Thue in 1906 [31]. He answered affirmatively the following two questions: Is there a binary sequence without cubes? Is there a ternary sequence without squares? In [32] he constructed the famous Thue–Morse sequence which is not only cube-free, but even overlap-free. Squares, cubes and overlaps are particular cases of *fractional powers*. A word w is a fractional power of a word u if an infinite repetition $uuu\cdots$ begins with w. The ratio of lengths of w and the shortest possible u is the *exponent* of w. The supremum of the exponents of all non-empty factors occurring in a sequence $\mathbf{u}$ is the *critical* (or *local*) exponent of $\mathbf{u}$. The critical exponent of sequences and related questions of repetition avoidance has become today a classic area of combinatorics on words.

Obviously, the larger the alphabet the smaller the critical exponent that can be found among the sequences over the alphabet. In 1988, Carpi [5] showed that for every real number $\alpha > 1$ there exists d and a d-ary pure morphic sequence with the critical exponent less than α. Krieger and Shallit [19] proved that every real number greater than 1 is a critical exponent of some sequence.

The search for the minimal critical exponent of infinite sequences over an alphabet of a fixed size resulted in a conjecture formulated by Dejean [11] in 1972. The conjecture states that the infimum of critical exponents of d-ary sequences equals:

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https://doi.org/10.1016/j.tcs.2022.04.021
0304-3975/© 2022 Published by Elsevier B.V.

- 2 for $d = 2$;
- $7/4$ for $d = 3$;
- $7/5$ for $d = 4$;
- $\frac{d}{d-1}$ for $d \geq 5$.

The conjecture had been proved step by step by many people [11,24,23,21,6,10,27].

The least critical exponent is also studied for particular families of sequences. Carpi and de Luca [7] found out that the minimal critical exponent of a Sturmian sequence is $\frac{5+\sqrt{5}}{2}$, reached by the Fibonacci sequence. The least critical exponent for binary rich sequences was determined recently by Curie, Mol and Rampersad [9]. The minimal value $2+\frac{\sqrt{2}}{2}$ is reached by a complementary symmetric Rote sequence. Shallit and Shur [29] proved a number of results connecting factor complexity and critical exponent of sequences. For example, they established that the Thue-Morse sequence and the twisted Thue-Morse sequence have, respectively, the minimum and the maximum factor complexity over all binary overlap-free sequences; the minimal critical exponent of a binary sequence of factor complexity $2n$ is $5/2$; the set of ternary square-free sequences either has no sequence of minimum complexity, or the minimum is reached by the ternary Thue sequence.

For some types of sequences, formulae for computation of the critical exponent are known. Blondin-Massé et al. [20] computed the critical exponent for generalized Thue-Morse sequences. The critical exponent of complementary symmetric Rote sequences is computed in [15]. Justin and Pirillo [17] gave a formula for the critical exponent of standard episturmian sequences which are fixed by a primitive morphism. Krieger [18] provided an algorithm to compute the critical exponent for sequences that are fixed points of non-erasing morphisms.

In this paper we focus on the least critical exponent of balanced sequences. Let us recall that a sequence over a finite alphabet is balanced if, for any two of its factors u and v of the same length, the number of occurrences of each letter in u and v differs by at most 1. Over a binary alphabet, aperiodic balanced sequences coincide with Sturmian sequences, as shown by Morse and Hedlund [22]. Hubert [16] provided a uniform construction of all d-ary aperiodic recurrent balanced sequences from Sturmian sequences. Recently, Rampersad, Shallit and Vandomme [26] found balanced sequences with the least critical exponent over alphabets of size 3 and 4 and also conjectured that the minimal critical exponent of balanced sequences over a d-ary alphabet with $d \geq 5$ is $\frac{d-2}{d-3}$. For $d \leq 10$, they defined the candidate sequences $\mathbf{x}_d$, obtained from Sturmian sequences with quadratic slope, to reach this minimum. In [12], an algorithm for computing the critical exponent of balanced sequences of this type is deduced. The conjecture of Rampersad, Shallit and Vandomme was confirmed for $d \leq 8$ in [2,3] and for $d \in \{9, 10\}$ in [12].

In this paper, we first show that for balanced sequences over a d-ary alphabet with $d \geq 11$, the critical exponent is greater than or equal to $\frac{d-1}{d-2}$. Then for every even $d \geq 12$, we find a d-ary balanced sequence $\mathbf{x}_d$ having the critical exponent $\frac{d-1}{d-2}$. Again, each sequence $\mathbf{x}_d$ is derived from a Sturmian sequence with a quadratic slope. In particular, for $d \geq 14$ this Sturmian sequence is the Fibonacci sequence.

As $\frac{d-1}{d-2} < \frac{d-2}{d-3}$, our result refutes the conjecture by Rampersad, Shallit and Vandomme. We state as a new conjecture that the minimal critical exponent of balanced sequences equals $\frac{d-1}{d-2}$ for $d \geq 11$. Thus it remains to prove this conjecture for sequences over alphabets of odd size.

2. Preliminaries

An *alphabet* $\mathcal{A}$ is a finite set of symbols called *letters*. A *word* over $\mathcal{A}$ of *length* n is a string $u = u_0u_1\cdots u_{n-1}$, where $u_i \in \mathcal{A}$ for all $i \in \{0, 1, \ldots, n-1\}$. The length of u is denoted by $|u|$. The set of all finite words over $\mathcal{A}$ together with the operation of concatenation forms a monoid, denoted $\mathcal{A}^*$. Its neutral element is the *empty word* ε and we denote $\mathcal{A}^+ = \mathcal{A}^* \setminus \{\varepsilon\}$. If $u = xyz$ for some $x, y, z \in \mathcal{A}^*$, then x is a *prefix* of u, z is a *suffix* of u and y is a *factor* of u. To any word u over $\mathcal{A}$ with cardinality $\#\mathcal{A} = d$, we assign its *Parikh vector* $\vec{V}(u) \in \mathbb{N}^d$ defined as $(\vec{V}(u))_a = |u|_a$ for all $a \in \mathcal{A}$, where $|u|_a$ is the number of letters a occurring in u.

A *sequence* over $\mathcal{A}$ is an infinite string $\mathbf{u} = u_0u_1u_2\cdots$, where $u_i \in \mathcal{A}$ for all $i \in \mathbb{N}$. The notation $\mathcal{A}^{\mathbb{N}}$ stands for the set of all sequences over $\mathcal{A}$. We always denote sequences by bold letters. The shift operator σ maps any sequence $\mathbf{u} = u_0u_1u_2\cdots$ to the sequence $\sigma(\mathbf{u}) = u_1u_2u_3\cdots$. The *frequency* of a letter a in a sequence $\mathbf{u}$ is the limit $\rho_a(\mathbf{u}) = \lim_{n\to\infty} \frac{|u_0\cdots u_{n-1}|_a}{n}$ if it exists.

A sequence $\mathbf{u}$ is *eventually periodic* if $\mathbf{u} = vwww\cdots = v(w)^\omega$ for some $v \in \mathcal{A}^*$ and $w \in \mathcal{A}^+$. It is *periodic* if $\mathbf{u} = w^\omega$. In both cases, the number $|w|$ is a *period* of $\mathbf{u}$. We write $\mathrm{Per}(\mathbf{u})$ for the minimal period of $\mathbf{u}$. If $\mathbf{u}$ is not eventually periodic, then it is *aperiodic*. A *factor* of $\mathbf{u} = u_0u_1u_2\cdots$ is a word y such that $y = u_iu_{i+1}u_{i+2}\cdots u_{j-1}$ for some $i, j \in \mathbb{N}$, $i \leq j$. The number i is called an *occurrence* of the factor y in $\mathbf{u}$. In particular, if $i = j$, the factor y is the empty word ε and any index i is its occurrence. If $i = 0$, the factor y is a *prefix* of $\mathbf{u}$. If each factor of $\mathbf{u}$ has infinitely many occurrences in $\mathbf{u}$, the sequence $\mathbf{u}$ is *recurrent*. Moreover, if for each factor the distances between its consecutive occurrences are bounded, $\mathbf{u}$ is *uniformly recurrent*. In a uniformly recurrent sequence all letters have frequencies.

The *language* $\mathcal{L}(\mathbf{u})$ of a sequence $\mathbf{u}$ is the set of all its factors. A factor w of $\mathbf{u}$ is *right special* if wa, wb are in $\mathcal{L}(\mathbf{u})$ for at least two distinct letters $a, b \in \mathcal{A}$. A *left special* factor is defined symmetrically. A factor is *bispecial* if it is both left and right special. The *factor complexity* of a sequence $\mathbf{u}$ is the mapping $\mathcal{C}_{\mathbf{u}} : \mathbb{N} \to \mathbb{N}$ defined by $\mathcal{C}_{\mathbf{u}}(n) = \#\{w \in \mathcal{L}(\mathbf{u}) : |w| = n\}$.

The factor complexity of an aperiodic sequence $\mathbf{u}$ satisfies $\mathcal{C}_{\mathbf{u}}(n) \geq n+1$ for all $n \in \mathbb{N}$. The aperiodic sequences with the lowest possible factor complexity $\mathcal{C}_{\mathbf{u}}(n) = n+1$ are called *Sturmian sequences*. Clearly, all Sturmian sequences are defined over a binary alphabet, e.g., $\{\mathtt{a}, \mathtt{b}\}$.

A sequence $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$ is *balanced* if for every letter $a \in \mathcal{A}$ and every pair of factors $u, v \in \mathcal{L}(\mathbf{u})$ with $|u| = |v|$, we have $|u|_a - |v|_a \leq 1$. The class of Sturmian sequences and the class of aperiodic balanced sequences coincide over a binary alphabet (see [22]). Every recurrent balanced sequence is uniformly recurrent (see [12]).

A *morphism* over $\mathcal{A}$ is a mapping $\psi : \mathcal{A}^* \to \mathcal{A}^*$ such that $\psi(uv) = \psi(u)\psi(v)$ for all $u, v \in \mathcal{A}^*$. Morphisms can be naturally extended to $\mathcal{A}^{\mathbb{N}}$ by setting $\psi(u_0u_1u_2\cdots) = \psi(u_0)\psi(u_1)\psi(u_2)\cdots$. A *fixed point* of a morphism ψ is a sequence $\mathbf{u}$ such that $\psi(\mathbf{u}) = \mathbf{u}$.

Consider a factor w of a recurrent sequence $\mathbf{u} = u_0u_1u_2\cdots$. Let $i < j$ be two consecutive occurrences of w in $\mathbf{u}$. Then the word $u_iu_{i+1}\cdots u_{j-1}$ is a *return word* to w in $\mathbf{u}$. The set of all return words to w in $\mathbf{u}$ is denoted by $\mathcal{R}_{\mathbf{u}}(w)$. If $\mathbf{u}$ is uniformly recurrent, the set $\mathcal{R}_{\mathbf{u}}(w)$ is finite for each factor w. If k is the first occurrence of a factor w in $\mathbf{u}$, then the k-th shift of $\mathbf{u}$ can be written as a concatenation $\sigma^k(\mathbf{u}) = r_{d_0}r_{d_1}r_{d_2}\cdots$ of return words to w. In particular, if w is a prefix of $\mathbf{u}$, then $\mathbf{u} = r_{d_0}r_{d_1}r_{d_2}\cdots$. The sequence $\mathbf{d}_{\mathbf{u}}(w) = d_0d_1d_2\cdots$ over the alphabet of cardinality $\#\mathcal{R}_{\mathbf{u}}(w)$ is called the *derived sequence* of $\mathbf{u}$ to w. The concept of derived sequences was introduced by Durand [14].

For an arbitrary nonempty word z, let u be the shortest word such that z is a prefix of the periodic sequence u^{ω}. The number $|u|$ is the (minimal) *period* of z, and the ratio $e = |z|/|u|$ is the *exponent* of z, written as $e = \exp(z)$. The *critical exponent* of an infinite sequence $\mathbf{u}$ is defined as

$$\mathrm{E}(\mathbf{u}) = \sup\{\exp(z) : z \text{ is a non-empty factor of } \mathbf{u}\}.$$

The critical exponent of a uniformly recurrent sequence[1] can be computed from its bispecial factors and their return words:

Theorem 1 *([13]). Let $\mathbf{u}$ be a uniformly recurrent aperiodic sequence. Let (w_n) be a sequence of all bispecial factors of $\mathbf{u}$, ordered by their length. For every $n \in \mathbb{N}$, let v_n be a shortest return word to w_n in $\mathbf{u}$. Then*

$$\mathrm{E}(\mathbf{u}) = 1 + \sup_{n\in\mathbb{N}} \left\{ \frac{|w_n|}{|v_n|} \right\}.$$

2.1. Sturmian sequences

Sturmian sequences are a principal tool in the study of balanced sequences over arbitrary alphabets. In this section we recall the necessary facts about them.

We recall that Sturmian sequences can be considered as *cutting sequences* of straight lines with irrational slopes [22]. The definition is as follows. Consider the positive quadrant of the coordinate plane and a square grid on it, parallel to the axes (the axes themselves do not belong to the grid). The intersection of a straight line with the grid can be encoded as a binary sequence: symbol $\mathtt{a}$ (resp., $\mathtt{b}$) encodes the intersection with the horizontal (resp., vertical) line of the grid. Thus, each Sturmian sequence $\mathbf{u}$ has an irrational *slope* $\theta(\mathbf{u})$ which is the slope of the straight line producing $\mathbf{u}$ as a cutting sequence. From the definition of the cutting sequence it is clear that the letter frequencies in a Sturmian sequence are irrational, and the slope of a Sturmian sequence equals the ratio of these frequencies.

All Sturmian sequences with the same slope share the same language. Among them, there is a unique *standard sequence*, which is the cutting sequence of a line intersecting the origin. Equivalently, the standard sequence can be defined by the condition that both sequences $\mathtt{a}\mathbf{u}$ and $\mathtt{b}\mathbf{u}$ are Sturmian.

Example 2. The most famous standard sequence is the *Fibonacci sequence*

$$\mathbf{u}_f = \mathtt{babbababbabbababbababb}\cdots,$$

defined as the fixed point of the morphism $f : \mathtt{b} \mapsto \mathtt{ba}$, $\mathtt{a} \mapsto \mathtt{b}$. Its slope is $\varphi = \frac{\sqrt{5}-1}{2} \approx 0.618$ (the inverse of the golden ratio), the frequencies of $\mathtt{a}$ and $\mathtt{b}$ are φ^2 and φ respectively, and the critical exponent of $\mathbf{u}_f$ is $3+\varphi$, which is the minimum among Sturmian sequences.

We use the characterization of standard sequences by their directive sequences. To introduce them, we define the two morphisms

$$G = \begin{cases} \mathtt{a} \to \mathtt{a} \\ \mathtt{b} \to \mathtt{ab} \end{cases} \quad \text{and} \quad D = \begin{cases} \mathtt{a} \to \mathtt{ba} \\ \mathtt{b} \to \mathtt{b} \end{cases}.$$

[1] In fact, the same result holds for all aperiodic recurrent sequences. As Theorem 1 is sufficient for our purposes, we do not prove the more general result here.

Proposition 3 *([17]). For every standard sequence* $\mathbf{u}$ *there is a uniquely given* directive sequence $\mathbf{\Delta} = \Delta_0\Delta_1\Delta_2\cdots \in \{G, D\}^{\mathbb{N}}$ *of morphisms and a sequence* $(\mathbf{u}^{(n)})$ *of standard sequences such that*

$$\mathbf{u} = \Delta_0\Delta_1 \ldots \Delta_{n-1}\left(\mathbf{u}^{(n)}\right) \ \textit{for every} \ n \in \mathbb{N}\,.$$

Both G and D occur in the sequence $\mathbf{\Delta}$ *infinitely often.*

If $\Delta_0 = D$, then by Proposition 3 $\mathbf{u}$ is the image of a standard sequence under the morphism D and consequently, $\mathtt{b}$ is the most frequent letter in $\mathbf{u}$. Otherwise, $\mathtt{a}$ is the most frequent letter in $\mathbf{u}$. We adopt the convention that $\rho_{\mathtt{b}}(\mathbf{u}) > \rho_{\mathtt{a}}(\mathbf{u})$ and thus the directive sequence of $\mathbf{u}$ starts with D. Let us write this sequence in the run-length encoded form $\mathbf{\Delta} = D^{a_1}G^{a_2}D^{a_3}G^{a_4}\cdots$, where all integers a_n are positive. Then the number θ having the continued fraction expansion $\theta = [0, a_1, a_2, a_3, \ldots]$ equals the ratio $\frac{\rho_{\mathtt{a}}(\mathbf{u})}{\rho_{\mathtt{b}}(\mathbf{u})}$ (see [4]) and thus $\theta = \theta(\mathbf{u})$.

Knowing the coefficients of the continued fraction of the slope of $\mathbf{u}$, one can find prefixes of $\mathbf{u}$. The more initial coefficients of $\theta(\mathbf{u})$ we have, the longer prefixes of $\mathbf{u}$ we can reconstruct. In particular, if the directive sequence of $\mathbf{u}$ starts with D^{a_1}, then $\mathbf{u} = \mathtt{b}^{a_1}\mathtt{a}\cdots$.

Example 4. Consider the standard sequence $\mathbf{u}$ with the slope $\theta = [0, 1, 3, \overline{2}]$. As any sequence of the form $G^2(\mathbf{w})$ starts with $\mathtt{aab}$, the following word is a prefix of $\mathbf{u}$:

$$\begin{aligned} DG^3D^2(\mathtt{aab}) &= DG^3(\mathtt{bbabbab}) = D(\mathtt{a}^3\mathtt{ba}^3\mathtt{ba}^4\mathtt{ba}^3\mathtt{ba}^4\mathtt{b}) = \\ &= \mathtt{bababab\,bababab\,babababab\,bababab\,babababab}. \end{aligned}$$

The convergents to the continued fraction of θ, usually denoted $\frac{p_N}{q_N}$, and their secondary convergents have a close relation to the return words in a Sturmian sequence. Recall that the sequences (p_N) and (q_N) both satisfy the recurrence relation

$$X_{N+1} = a_{N+1}X_N + X_{N-1} \tag{1}$$

with initial conditions $p_{-1} = 1$, $p_0 = 0$ and $q_{-1} = 0$, $q_0 = 1$. Two consecutive convergents satisfy $p_Nq_{N-1} - p_{N-1}q_N = (-1)^{N+1}$ for every $N \in \mathbb{N}$.

Vuillon [33] showed that an infinite recurrent sequence $\mathbf{u}$ is Sturmian if and only if each of its factors has exactly two return words. Moreover, the derived sequence of a Sturmian sequence to any of its factors is also Sturmian.

All bispecial factors of any standard sequence $\mathbf{u}$ are prefixes of $\mathbf{u}$. So, one of the return words to a bispecial factor of $\mathbf{u}$ is a prefix of $\mathbf{u}$.

Proposition 5 *([15]). Suppose that* $\mathbf{u}$ *is a standard sequence with the slope* $\theta = [0, a_1, a_2, a_3, \ldots]$ *and z is a bispecial factor of* $\mathbf{u}$*. Let r (resp., s) denote the return word to z which is (resp., is not) a prefix of* $\mathbf{u}$*. Then*

1. *there exists a unique pair* $(N, m) \in \mathbb{N}^2$ *with* $0 \le m < a_{N+1}$ *such that the Parikh vectors of r, s, and z are respectively*
$$\vec{V}(r) = \begin{pmatrix} p_N \\ q_N \end{pmatrix}, \quad \vec{V}(s) = \begin{pmatrix} m\,p_N + p_{N-1} \\ m\,q_N + q_{N-1} \end{pmatrix}, \quad \vec{V}(z) = \vec{V}(r) + \vec{V}(s) - \begin{pmatrix} 1 \\ 1 \end{pmatrix};$$
2. *the slope of the derived sequence* $\mathbf{d_u}(z)$ *is*
$$\theta' = [0, a_{N+1} - m, a_{N+2}, a_{N+3}, \ldots].$$

Lemma 6 *([13]). Let* $\mathbf{u}$ *be a Sturmian sequence with the slope* $\theta = \frac{\rho_{\mathtt{a}}(\mathbf{u})}{\rho_{\mathtt{b}}(\mathbf{u})} < 1$*. Then* $\mathbf{u}$ *contains a factor w such that* $|w|_{\mathtt{b}} = k$ *and* $|w|_{\mathtt{a}} = \ell$ *if and only if*

$$(k-1)\theta - 1 < \ell < (k+1)\theta + 1 \ \textit{and}\ k, \ell \in \mathbb{N}. \tag{2}$$

2.2. Balanced sequences

In 2000 Hubert [16] characterized balanced sequences over alphabets of cardinality bigger than 2 in terms of Sturmian sequences, *colourings*, and *constant gap sequences.*

Definition 7. Let $\mathbf{u}$ be a sequence over $\{\mathtt{a}, \mathtt{b}\}$, $\mathbf{y}$ and $\mathbf{y}'$ be arbitrary sequences. The *colouring*[2] of $\mathbf{u}$ by $\mathbf{y}$ and $\mathbf{y}'$ is the sequence $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$ obtained from $\mathbf{u}$ by replacing the subsequence of all $\mathtt{a}$'s with $\mathbf{y}$ and the subsequence of all $\mathtt{b}$'s with $\mathbf{y}'$.

[2] This operation is also known as *shuffling* of $\mathbf{y}$ and $\mathbf{y}'$ with *directive sequence* $\mathbf{u}$.

Definition 8. A sequence $\mathbf{y}$ is a *constant gap sequence* if for each letter a occurring in $\mathbf{y}$ there is a positive integer denoted by $\mathrm{gap}_{\mathbf{y}}(a)$ such that the distance between any consecutive occurrences of a in $\mathbf{y}$ is $\mathrm{gap}_{\mathbf{y}}(a)$.

Obviously, every constant gap sequence $\mathbf{y}$ is periodic and $\mathrm{Per}(\mathbf{y})$ is the least common multiple of all numbers $\mathrm{gap}_{\mathbf{y}}(a)$.

Example 9. The sequence $\mathbf{y} = (\mathtt{0102})^{\omega}$ is a constant gap sequence because the distance between consecutive 0's is always 2, while the distance between consecutive 1's (resp., 2's) is always 4. Its minimal period is $\mathrm{Per}(\mathbf{y}) = 4$.

The sequence $\mathbf{v} = \mathrm{colour}(\mathbf{u}_f, (\mathtt{AB})^{\omega}, (\mathtt{0102})^{\omega})$, where $\mathbf{u}_f$ is defined in Example 2, looks as follows:

$$\mathbf{u}_f = \mathtt{babbababbabbababbababbabbababba}\cdots$$
$$\mathbf{v} = \mathtt{0A10B2A01B02A0B10A2B01A02B0A10B}\cdots$$

Theorem 10 *([16]). A recurrent aperiodic sequence* $\mathbf{v}$ *is balanced if and only if* $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$ *for some Sturmian sequence* $\mathbf{u}$ *and constant gap sequences* $\mathbf{y}, \mathbf{y}'$ *over two disjoint alphabets.*

Corollary 11. *A letter a from $\mathbf{y}$ (resp., b from $\mathbf{y}'$) occurs in $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$ with frequency $\rho_a(\mathbf{v}) = \frac{\rho_{\mathtt{a}}(\mathbf{u})}{\mathrm{gap}_{\mathbf{y}}(a)}$ (resp., $\rho_b(\mathbf{v}) = \frac{\rho_{\mathtt{b}}(\mathbf{u})}{\mathrm{gap}_{\mathbf{y}'}(b)}$). In particular, all frequencies of letters in aperiodic balanced sequences are irrational.*

Example 9 *(continued).* By Theorem 10, the sequence $\mathbf{v}$ is balanced. Knowing the frequencies of letters in $\mathbf{u}_f$ from Example 2, we can compute the frequencies of letters in $\mathbf{v}$ by Corollary 11; e.g., $\rho_0(\mathbf{v}) = \varphi/2$.

Let $\mathcal{A}, \mathcal{B}$ be two disjoint alphabets. The "discolouration map" π is defined for any word or sequence over $\mathcal{A} \cup \mathcal{B}$; it replaces all letters from $\mathcal{A}$ by $\mathtt{a}$ and all letters from $\mathcal{B}$ by $\mathtt{b}$. If $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$, where $y \in \mathcal{A}^{\mathbb{N}}$, $y' \in \mathcal{B}^{\mathbb{N}}$, then $\pi(\mathbf{v}) = \mathbf{u}$ and $\pi(v) \in \mathcal{L}(\mathbf{u})$ for every $v \in \mathcal{L}(\mathbf{v})$.

3. Lower bounds on critical exponent

First we prove a simple but useful property.

Lemma 12. *Let* $\mathbf{v}$ *be an aperiodic balanced sequence, a be a letter in* $\mathbf{v}$*. The set of distances between consecutive occurrences of a in* $\mathbf{v}$ *consists of two consecutive integers.*

Proof. Since the frequency of a in $\mathbf{v}$ is irrational by Corollary 11, there should be at least two different distances between consecutive occurrences of a. Let k be the minimal such distance. Then $\mathbf{v}$ has a factor of length $k+1$ with two a's. By the balance property, each factor of this length contains the letter a, so the distance between consecutive a's cannot exceed $k+1$. Hence the set of distances is $\{k, k+1\}$, as required. □

Now we state the main result of this section.

Theorem 13. *For each $d \geq 11$, there exists no d-ary balanced sequence* $\mathbf{v}$ *with* $\mathrm{E}(\mathbf{v}) < \frac{d-1}{d-2}$.

Let us fix an arbitrary alphabet $\mathcal{A}$ with $d \geq 11$ letters and introduce the necessary tools.[3] Any sequence $\mathbf{v} \in \mathcal{A}^{\mathbb{N}}$ with $\mathrm{E}(\mathbf{v}) < \frac{d-1}{d-2}$ satisfies the following "local" properties:

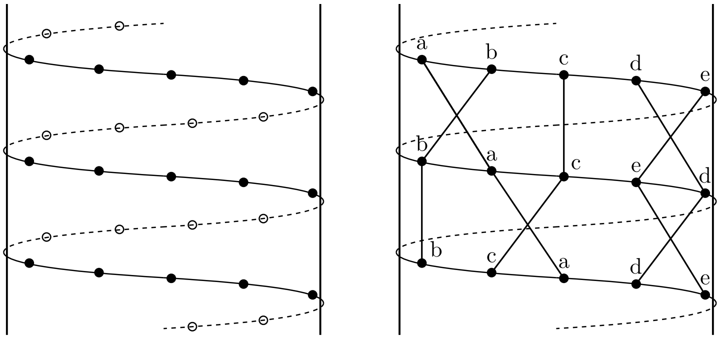
(i) in each factor of $\mathbf{v}$ of length $d-1$, all letters are distinct;
(ii) any two consecutive occurrences of the same letter in $\mathbf{v}$ are followed by different letters.

Property (i) is obvious. If (ii) fails, then $\mathbf{v}$ has a factor $abXab$, where the word X does not contain the letter a. Assume X contains b: $X = YbZ$. By (i), $|bYb|, |bZab| \geq d$. Then $|bX| \geq 2d-3$, and bX contains only $d-1$ distinct letters. Again by (i), bX has the period $d-1$ and thus $\exp(bX) \geq \frac{2d-3}{d-1}$, which is impossible. Therefore X contains neither of the letters a, b. Then $|X| \leq d-2$ by (i), implying

$$\mathrm{E}(\mathbf{v}) \geq \exp(abXab) = \frac{|abXab|}{|abX|} \geq \frac{d+2}{d} > \frac{d-1}{d-2}, \tag{3}$$

which is impossible. Thus, (ii) holds.

[3] The argument below works for $d \geq 4$, but since for $d \leq 10$ a stronger lower bound is known [26], only the case $d \geq 11$ is of interest.



(a) Word on a cylinder ($d = 9$) (b) Sticks and traces (only visible)

Fig. 1. Cylindric representation of a Pansiot word.

Following [30], we refer to the words and sequences satisfying (i) and (ii) as *Pansiot words/sequences*. Note that (i) and (ii) imply that a factor of length $d+1$ should contain d distinct letters. As a result, Pansiot words/sequences satisfy the property

(iii) any two consecutive occurrences of the same letter are at the distance $d-1$, d, or $d+1$.

A handy tool for studying Pansiot words and sequences is the *cylindric representation* introduced in [30]. A Pansiot word/sequence can be viewed as a rope with knots representing letters. This rope is wound around a cylinder such that the knots at distance d are placed one under another. A part of a projection of such a cylinder is drawn in Fig. 1, a. By (iii), the knots labelled by two consecutive occurrences of the same letter appear on two consecutive winds of the rope one under another or shifted by one knot (Fig. 1, b). Connecting consecutive occurrences by line segments ("sticks"), we get three types of sticks: vertical, left-slanted, and right-slanted (Fig. 1, b); these three types correspond respectively to the distances d, $d+1$, and $d-1$ between the occurrences. The sticks form d broken lines, one per letter; we call these lines *traces* of letters.

From (ii), we derive the following property:

(iv) any two subsequent sticks in the cylindric representation of a Pansiot word (sequence) are distinct.

From (iii) and Lemma 12 we have:

($\star$) the trace of a letter in the cylindric representation of a balanced Pansiot sequence contains either no left-slanted sticks or no right-slanted sticks.

(For example, if the cylindric representation of $\mathbf{v}$ contains a fragment shown in Fig. 1, b, then $\mathbf{v}$ is not balanced.) We call a letter *frequent* (in $\mathbf{v}$) if it has no consecutive occurrences at distance $d+1$ and *rare* (in $\mathbf{v}$) if it has no consecutive occurrences at distance $d-1$. The following key lemma shows that the distance d is possible only for one of these two classes of letters.

Lemma 14. *In a balanced Pansiot sequence, either the traces of all frequent letters consist only of right-slanted sticks or the traces of all rare letters consist only of left-slanted sticks.*

Proof. Let us fix a balanced Pansiot sequence $\mathbf{v} = v_0v_1v_2\cdots$ and assume that the trace of some frequent letter a in $\mathbf{v}$ contains a vertical stick (otherwise, there is nothing to prove). Consider a fragment of the cylindric representation of $\mathbf{v}$ around a fixed vertical stick in the trace of a (Fig. 2, a–d). By (iv), this stick (the blue one) is surrounded by pairs of crossed slanted sticks (Fig. 2, a). Note that two vertical sticks cannot have a common knot: this contradicts ($\star$) (Fig. 2, b). Since a is frequent, two more crossed pairs should be added to the picture (Fig. 2, c). Next, the traces of the letters c and d contain right-slanted sticks, so these letters are frequent. Then the trace of c (resp., of d) extends up (resp., down) by a vertical stick (Fig. 2, d). Thus we proved the following fact: if, for some i, $v_i = v_{i+d}$ is a frequent letter, then $v_{i-d-1} = v_{i-1}$ is a frequent letter and $v_{i+d+1} = v_{i+2d+1}$ is also a frequent letter (these three pairs of equal letters correspond to three vertical sticks in Fig. 2, d). Thus

(†) there exists $p \in \{0, \ldots, d\}$ such that all positions equal to p modulo $d+1$ are occupied in $\mathbf{v}$ by frequent letters and correspond to vertical sticks.

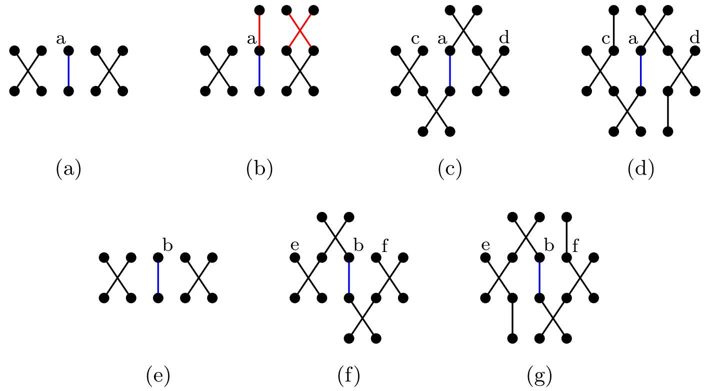


Fig. 2. Lemma 14: mutual location of vertical sticks in the traces of frequent and rare letters. (For interpretation of the colours in the figure(s), the reader is referred to the web version of this article.)

Now assume that the trace of some rare letter b in $\mathbf{v}$ contains a vertical stick (or there is nothing to prove). By the same argument as above, we reconstruct a fragment of the cylindric representation of $\mathbf{v}$ (Fig. 2, e-g). As a result, we get the following analog of (†):

(‡) there exists $q \in \{0, \dots, d-2\}$ such that all positions equal to q modulo $d-1$ are occupied in $\mathbf{v}$ by rare letters and correspond to vertical sticks.

It remains to note that (†) and (‡) cannot hold simultaneously. Indeed, if p and q have the same parity, then some letter should be both frequent and rare in $\mathbf{v}$, which is impossible (see Fig. 2, b); if p and q are of different parity, then the cylindric representation of $\mathbf{v}$ contains two consecutive vertical sticks, which contradicts (ii). But as (†) and (‡) does not hold simultaneously, at least one of our assumptions about the existence of vertical sticks is false. The statement of the lemma is immediate from this. $\square$

Proof of Theorem 13. Let us take a d-ary aperiodic balanced sequence $\mathbf{v}$ and assume $\mathrm{E}(\mathbf{v}) < \frac{d-1}{d-2}$. Then $\mathbf{v}$ is a Pansiot sequence. By Lemma 14, either each frequent letter in $\mathbf{v}$ has frequency $\frac{1}{d-1}$ or each rare letter in $\mathbf{v}$ has frequency $\frac{1}{d+1}$. But these frequencies must be irrational by Corollary 11. This contradiction proves our assumption false. $\square$

If the bound of Theorem 13 is tight, then the balanced sequences of minimal critical exponent should satisfy the following property.

Proposition 15. *Let $\mathbf{v}$ be a balanced sequence over $d \geq 11$ letters such that $\mathrm{E}(\mathbf{v}) = \frac{d-1}{d-2}$. Then the maximum letter frequency ρ in $\mathbf{v}$ satisfies $\frac{1}{d-1} < \rho < \frac{1}{d-2}$.*

Proof. Every factor of length $d-2$ in $\mathbf{v}$ contains no repeating letters. Hence the minimal distance between two occurrences of a letter is $d-2$. Assume that every factor of length $d-1$ in $\mathbf{v}$ contains no repeating letters. Then $\mathbf{v}$ is a Pansiot sequence: it satisfies both (i) and (ii) (the negation of (ii) implies the inequality (3), contradicting the condition $\mathrm{E}(\mathbf{v}) = \frac{d-1}{d-2}$). As was shown in the proof of Theorem 13, there are no aperiodic balanced Pansiot sequences, so our assumption is false. Therefore, $\mathbf{v}$ contains factors of length $d-1$, having the form aXa. By Lemma 12, the distances between consecutive a's in $\mathbf{v}$ are $d-2$ and $d-1$. As the frequency of a is irrational, the inequalities $\frac{1}{d-1} < \rho_a(\mathbf{v}) < \frac{1}{d-2}$ are strict. $\square$

4. Balanced sequences reaching the lower bound

In this section we show that the lower bound $\frac{d-1}{d-2}$ is attained for balanced sequences over a d-ary alphabet with $d = 2\delta$ and $\delta \geq 6$. For each alphabet, the considered sequence $\mathbf{x}_{2\delta}$ is the colouring by $\mathbf{y} = (12\cdots\delta)^\omega$ and $\mathbf{y}' = (1'2'\cdots\delta')^\omega$ of a standard sequence with the slope θ that will be specified later. We treat the cases $\delta \geq 7$ and $\delta = 6$ separately, but we begin with the statements that are common to both cases. By Theorem 1, to prove that $E(\mathbf{x}_{2\delta}) = \frac{d-1}{d-2}$ it is sufficient to show that every bispecial factor $w \in \mathcal{L}(\mathbf{x}_{2\delta})$ and its shortest return word v satisfy $\frac{|w|}{|v|} \leq \frac{1}{2\delta-2}$.

Let $\mathbf{x}_{2\delta} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$. There are only two distinct frequencies of letters in $\mathbf{x}_{2\delta}$, namely, $\frac{\rho_{\mathrm{a}}(\mathbf{u})}{\delta}$ and $\frac{\rho_{\mathrm{b}}(\mathbf{u})}{\delta}$. By Proposition 15, $\frac{1}{2\delta-1} < \frac{\rho_{\mathrm{b}}(\mathbf{u})}{\delta} < \frac{1}{2\delta-2}$. This can be converted into the double inequality $1 - \frac{2}{\delta} < \theta < 1 - \frac{1}{\delta}$ for the slope of $\mathbf{u}$ (recall that $\theta = \frac{\rho_{\mathrm{a}}(\mathbf{u})}{\rho_{\mathrm{b}}(\mathbf{u})}$). Given this restriction, we define the slope to have the form $\theta = [0, 1, \lfloor \delta/2 \rfloor, \dots]$. Such a slope guarantees that $\mathbf{x}_{2\delta}$ contains no short factors with the exponent greater than $\frac{d-1}{d-2}$, as the following proposition shows.

Proposition 16. *Suppose that $\mathbf{x}_{2\delta} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$, where $\mathbf{y} = (\mathtt{1}\mathtt{2}\ldots\delta)^\omega$, $\mathbf{y}' = (\mathtt{1}'\mathtt{2}'\cdots\delta')^\omega$, and the slope of the Sturmian sequence $\mathbf{u}$ has the form $\theta = [0, 1, \lfloor\delta/2\rfloor, \ldots]$. Then*

1. *the distance between occurrences of a letter in $\mathbf{x}_{2\delta}$ is at least $2\delta - 2$;*
2. *the distance between occurrences of a length-2 factor, both letters of which are from $\mathbf{y}'$, in $\mathbf{x}_{2\delta}$, is greater than $4\delta - 4$.*

Proof. The given prefix of the continued fraction expansion of θ allows us to conclude that $\mathbf{u}$ is an infinite concatenation of the "blocks" $(\mathtt{ba})^{\lfloor\delta/2\rfloor}\mathtt{b}$ and $(\mathtt{ba})^{\lfloor\delta/2\rfloor+1}\mathtt{b}$ (cf. Example 4).

Let us fix an arbitrary letter a and consider a factor in $\mathbf{x}_{2\delta}$ of minimal length containing two a's: $v = aXa$. Let $u = \pi(v)$; statement 1 is thus equivalent to $|u| \geq 2\delta - 1$. The structure of the sequences $\mathbf{y}$ and $\mathbf{y}'$ implies that u contains $\delta + 1$ occurrences of the letter $\pi(a)$. Since $\mathbf{u}$ has no factor $\mathtt{aa}$, $\pi(a) = \mathtt{a}$ implies $|u| \geq 2\delta + 1$. Now let $\pi(a) = \mathtt{b}$. Note that the number of $\mathtt{b}$'s in two consecutive blocks is at least $2\lfloor\delta/2\rfloor + 2 \geq \delta + 1$. Since $|u|_\mathtt{b} = \delta + 1$, the factor u intersects at most three blocks in $\mathbf{u}$. In a factor of a block, the number of $\mathtt{b}$'s exceeds the number of $\mathtt{a}$'s by at most 1. Hence, $|u|_\mathtt{b} - |u|_\mathtt{a} \leq 3$, yielding $|u| \geq 2\delta - 1$. Statement 1 is proved.

Now let $v = abXab$ be a shortest factor in $\mathbf{x}_{2\delta}$ with the property $\pi(ab) = \mathtt{bb}$. Let $u = \pi(v)$. The structure of $\mathbf{y}$ and $\mathbf{y}'$ implies $|u|_\mathtt{b} = c\delta + 2$ for some $c \in \mathbb{N}$. The location of the factors $\mathtt{bb}$ in u implies that $u = \mathtt{b}Z\mathtt{b}$, where Z is a concatenation of blocks. Statement 2 is equivalent to $|Z| > 4\delta - 4$. We have $|Z|_\mathtt{b} = c\delta$, while each block contains either $\lfloor\delta/2\rfloor + 1$ or $\lfloor\delta/2\rfloor + 2$ $\mathtt{b}$'s. Since $\delta \geq 6$, the number of $\mathtt{b}$'s in one block is less than δ while the number of $\mathtt{b}$'s in two blocks is greater than δ. Hence $c \geq 2$. If $c = 2$, then Z consists of exactly three blocks, implying $|Z|_\mathtt{a} = |Z|_\mathtt{b} - 3 = 2\delta - 3$ and thus $|Z| = 4\delta - 3$. If $c > 2$, $|Z| > 4\delta - 4$ holds trivially. Statement 2 is proved. □

Let us recall that the length $|v|$ of a return word equals the difference between two consecutive occurrences of w in $\mathbf{x}_{2\delta}$. Furthermore, if an index i is an occurrence of w in $\mathbf{x}_{2\delta}$, then i is an occurrence of $\pi(w)$ in $\mathbf{u}$. Hence, $\pi(v)$ is a concatenation of return words to $\pi(w)$ in $\mathbf{u}$. Thus if r and s are the return words to $\pi(w)$ in $\mathbf{u}$, then

$$\text{there exist } k, \ell \in \mathbb{N}, k + \ell \geq 1 \ \text{ such that } \ |v| = |\pi(v)| = k|r| + \ell|s|. \tag{4}$$

In the sequel, if $\pi(w)$ is bispecial, we always assume that r is the prefix return word to $\pi(w)$ and s is the non-prefix one.

Of course, not all pairs (k, ℓ) correspond to a concatenation of r and s forming $\pi(v)$. The possible combinations are given by factors of the derived sequence of $\mathbf{u}$ to $\pi(w)$. Formally,

$$\begin{pmatrix} \ell \\ k \end{pmatrix} \text{ is the Parikh vector of a factor in } \ \mathbf{d}_\mathbf{u}(\pi(w)). \tag{5}$$

The simple form of $\mathbf{y}$ and $\mathbf{y}'$ implies some evident properties.

Lemma 17. *Let v be a return word in $\mathbf{x}_{2\delta}$ to a non-empty factor w such that $\pi(w)$ contains both $\mathtt{a}$ and $\mathtt{b}$.*

1. *Both $|\pi(v)|_\mathtt{a}$ and $|\pi(v)|_\mathtt{b}$ are divisible by δ.*
2. *If w is a bispecial factor of $\mathbf{x}_{2\delta}$, then $\pi(w)$ is a bispecial factor of $\mathbf{u}$.*

Item 2 of Lemma 17 enables us to exploit Proposition 5 on bispecial factors and return words in Sturmian sequences.

Lemma 18. *Let w be a bispecial factor of $\mathbf{x}_{2\delta}$ such that $\pi(w)$ contains both $\mathtt{a}$ and $\mathtt{b}$ and let θ' be the slope of the derived sequence $\mathbf{d}_\mathbf{u}(\pi(w))$. Then for every return word v to w in $\mathbf{x}_{2\delta}$, there exist k and ℓ such that*

1. $k, \ell \in \mathbb{N}, k + \ell \geq 1$,
2. $k = 0 \mod \delta, \ \ \ell = 0 \mod \delta$,
3. $\theta'(k-1) - 1 < \ell < \theta'(k+1) + 1$

and $|v| = k|r| + \ell|s|$, where r is the prefix and s the non-prefix return word to $\pi(w)$ in $\mathbf{u}$.

Proof. We need to show that k, ℓ in Equation (4) have also the properties described in Items 2 and 3. Let us denote by A the matrix from $\mathbb{N}^{2\times 2}$ such that $\vec{V}(s)$ and $\vec{V}(r)$ are the first and the second column of A, respectively. Then $\vec{V}(\pi(v)) = A\begin{pmatrix} \ell \\ k \end{pmatrix}$. By Item 1 of Lemma 17, $A\begin{pmatrix} \ell \\ k \end{pmatrix} = \delta\begin{pmatrix} L \\ K \end{pmatrix}$ for some integers L, K. By Proposition 5, $\det A = \pm 1$, and thus the inverse matrix A^{-1} belongs to $\mathbb{Z}^{2\times 2}$. Hence $\begin{pmatrix} \ell \\ k \end{pmatrix} = \delta A^{-1}\begin{pmatrix} L \\ K \end{pmatrix}$ and Item 2 follows.

Item 3 is a direct consequence of (5) and Lemma 6. □

4.1. Balanced sequence over d letters with $d \geq 14$, d even

Theorem 19. *Let* $\mathbf{u}$ *be the standard sequence with the slope* $\theta = [0, 1, \lfloor \delta/2 \rfloor, \overline{1}]$, $\delta \geq 7$ *be an integer,* $\mathbf{y} = (12\cdots\delta)^\omega$, $\mathbf{y}' = (1'2'\cdots\delta')^\omega$. *Then the balanced sequence* $\mathbf{x}_{2\delta} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$ *over the alphabet of* 2δ *letters has the critical exponent* $\frac{2\delta-1}{2\delta-2}$.

The next proposition is crucial in proving Theorem 19.

Proposition 20. *Let* w *be a bispecial factor of* $\mathbf{x}_{2\delta}$ *such that* $\pi(w)$ *contains* $\mathtt{a}$ *and* $\mathtt{b}$. *Then* $\frac{|w|}{|v|} < \frac{1}{2\delta-2}$ *for every return word* v *to* w *in* $\mathbf{x}_{2\delta}$.

Proof. By Lemma 17, $z := \pi(w)$ is a bispecial factor in $\mathbf{u}$. As z contains both $\mathtt{a}$ and $\mathtt{b}$, it can be any bispecial factor of $\mathbf{u}$ except $\mathtt{b}$. We use Proposition 5 to find the pair (N, m) with $m < a_{N+1}$ corresponding to z and the slope θ' of the derived sequence $\mathbf{d_u}(z)$. By the definition of convergents, the pair $(1, 0)$ corresponds to the factor $\mathtt{b}$. So we need to analyse all pairs $(N, 0)$ for $N \geq 2$ and all pairs $(1, m)$ for $m \in \{1, \ldots, \lfloor \delta/2 \rfloor - 1\}$. Recall that $\varphi = \frac{\sqrt{5}-1}{2}$ is the slope of the Fibonacci sequence.

- The pair $(N, 0)$, $N \geq 2$: By Proposition 5, $\theta' = [0, \overline{1}] = \varphi$.
  Let k, ℓ satisfy Items 1–3 from Lemma 18. Then $k = \delta k'$ and $\ell = \delta \ell'$ for some $\ell', k' \in \mathbb{N}, k' + \ell' \geq 1$. Dividing all parts of Item 3 by δ, we get
  $$\varphi k' - \frac{\varphi+1}{\delta} < \ell' < \varphi k' + \frac{\varphi+1}{\delta}. \tag{6}$$
  As $\delta \geq 7$, the left inequality fails for $\ell' = 0$, $k' > 0$ and for $\ell' = 1$, $k' > 1$; the right inequality does not hold for $\ell' = k' = 1$. Hence $\ell' \geq 2$ and $k' \geq 3$. Then $|v| = \delta k'|r| + \delta \ell'|s| \geq 3\delta|r| + 2\delta|s|$. By Proposition 5, $|w| = |\pi(w)| = |r| + |s| - 2$. Thus
  $$\frac{|w|}{|v|} \leq \frac{|r|+|s|-2}{3\delta|r|+2\delta|s|} < \frac{1}{2\delta} < \frac{1}{2\delta-2}. \tag{7}$$
- The pair $(1, m)$ with $m \in \{1, \ldots, \lfloor \delta/2 \rfloor - 1\}$: Using (1) and Proposition 5, we get $|r| = p_1 + q_1 = 2$, $|s| = m(p_1 + q_1) + p_0 + q_0 = 2m + 1$ and $\theta' = [0, \lfloor \delta/2 \rfloor - m, \overline{1}] = \frac{1}{\lfloor \delta/2 \rfloor - m + \varphi} \leq \frac{1}{1+\varphi} = \varphi$.
  Let k, ℓ satisfy Items 1–3 from Lemma 18. Then $k = \delta k'$ and $\ell = \delta \ell'$ for some $\ell', k' \in \mathbb{N}, k' + \ell' \geq 1$. Similar to (6), we get
  $$\theta' k' - \frac{\theta'+1}{\delta} < \ell' < \theta' k' + \frac{\theta'+1}{\delta}. \tag{8}$$
  As $\theta' > \frac{2}{\delta}$, the lefthand part is positive for $k' \neq 0$, implying $\ell' > 0$. As $\theta' \leq \varphi$, the right inequality holds only if $\ell' < k'$. Thus if $\ell' \geq 2$, then we bound the ratio $\frac{|w|}{|v|}$ as in (7). It remains to study the case $\ell' = 1$. Substituting the values of ℓ' and θ' into (8), we get the following double inequality for k':
  $$\left\lfloor \frac{\delta}{2} \right\rfloor - m + \varphi - \frac{\lfloor \frac{\delta}{2} \rfloor - m + 1 + \varphi}{\delta} < k' < \left\lfloor \frac{\delta}{2} \right\rfloor - m + \varphi + \frac{\lfloor \frac{\delta}{2} \rfloor - m + 1 + \varphi}{\delta}. \tag{9}$$
  Since $\delta \geq 7$ and $1 \leq m < \lfloor \delta/2 \rfloor$, we have $\varphi > \frac{\lfloor \delta/2 \rfloor - m + 1 + \varphi}{\delta}$. Hence the only integer solution of (9) is $k' = \lfloor \delta/2 \rfloor - m + 1$. The fact that it is a solution also restricts m through the inequality $\varphi + \frac{\lfloor \delta/2 \rfloor - m + 1 + \varphi}{\delta} > 1$, which transforms to
  $$m < \varphi(\delta+1) + \lfloor \delta/2 \rfloor + 1 - \delta. \tag{10}$$
  Estimating the ratio $|w|/|v|$, we get
  $$\frac{|w|}{|v|} \leq \frac{|r|+|s|-2}{k'\delta|r| + \delta|s|} = \frac{2 + (2m+1) - 2}{(\lfloor \delta/2 \rfloor - m + 1) \cdot \delta \cdot 2 + \delta \cdot (2m+1)} = \frac{2m+1}{\delta(2\lfloor \delta/2 \rfloor + 3)}. \tag{11}$$
  To finish the proof, it remains to verify that
  $$\frac{2m+1}{\delta(2\lfloor \delta/2 \rfloor + 3)} < \frac{1}{2\delta-2}. \tag{12}$$
  From (10) we have $m = 1$ for $\delta = 7$ and $m \leq 2$ for $\delta \in \{8, 9, 10\}$. So in all these cases (12) trivially holds. If $\delta \geq 11$, the right hand side of (10) satisfies
  $$\varphi(\delta+1) + \lfloor \delta/2 \rfloor + 1 - \delta \leq (\varphi - \tfrac{1}{2})\delta + 1 + \varphi < \tfrac{1}{4}\delta + \tfrac{1}{4},$$
  yielding
  $$\frac{2m+1}{\delta(2\lfloor \delta/2 \rfloor + 3)} < \frac{2\frac{\delta+1}{4} + 1}{\delta(2\lfloor \delta/2 \rfloor + 3)} \leq \frac{\delta+3}{2\delta(\delta+2)} < \frac{1}{2\delta-2}. \quad \square$$

Proof of Theorem 19. By Theorem 1, it is sufficient to show that $\frac{|w|}{|v|} \le \frac{1}{2\delta-2}$ for every bispecial factor w of $\mathbf{x}_{2\delta}$ and every return word v to w in $\mathbf{x}_{2\delta}$. If $\pi(w)$ contains both $\mathtt{a}$ and $\mathtt{b}$, the required inequality follows from Proposition 20. If $\pi(w)$ contains only one of these letters, the inequality follows from Proposition 16. $\square$

Remark 21. The inequalities in Proposition 20 and in Item 2 of Proposition 16 are strict, so the only type of factor of exponent $\frac{2\delta-1}{2\delta-2}$ in $\mathbf{x}_{2\delta}$ is the repeat of a single letter at distance $2\delta-2$. As Proposition 15 shows, such repeats are unavoidable in balanced d-ary sequences with the critical exponent $\frac{d-1}{d-2}$.

Remark 22. The only step in the proof of Proposition 20 that fails for $\delta = 6$ is the derivation from inequality (6): the case $\ell' = 1$, $k' = 2$ becomes possible. Indeed, for $\delta = 6$, the slope is $\theta = [0, 1, 3, \overline{1}]$. By Proposition 5, the Parikh vectors of the return words r and s to the bispecial factor corresponding to the pair $(7, 0)$ are $\vec{V}(r) = \begin{pmatrix} p_7 \\ q_7 \end{pmatrix} = \begin{pmatrix} 29 \\ 37 \end{pmatrix}$ and $\vec{V}(s) = \begin{pmatrix} p_6 \\ q_6 \end{pmatrix} = \begin{pmatrix} 18 \\ 23 \end{pmatrix}$. Then

$$\frac{|w|}{|v|} = \frac{|r| + |s| - 2}{k'\delta|r| + \ell'\delta|s|} = \frac{66 + 41 - 2}{2 \cdot 6 \cdot 66 + 1 \cdot 6 \cdot 41} = \frac{105}{1038} > \frac{1}{10}\,.$$

In fact, 1038 is the minimal period of a factor with the exponent $> \frac{11}{10}$.

4.2. Balanced sequence over 12 letters

Theorem 23. *Let* $\mathbf{u}$ *be the standard sequence with the slope* $\theta = [0, 1, 3, \overline{2}]$, $\mathbf{y} = (123456)^{\omega}$, $\mathbf{y}' = (1'2'3'4'5'6')^{\omega}$. *Then the critical exponent of the balanced sequence* $\mathbf{x}_{12} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$ *is* $\frac{11}{10}$.

We prove the following analog of Proposition 20.

Proposition 24. *Let* w *be a bispecial factor of* $\mathbf{x}_{12}$ *such that* $\pi(w)$ *contains* $\mathtt{a}$ *and* $\mathtt{b}$. *Then* $\frac{|w|}{|v|} < \frac{1}{10}$ *for every return word* v *to* w *in* $\mathbf{x}_{12}$.

Proof. By Lemma 17, $z := \pi(w)$ is a bispecial factor in $\mathbf{u}$. As z contains both $\mathtt{a}$ and $\mathtt{b}$, it can be any bispecial factor of $\mathbf{u}$ except $\mathtt{b}$. We use Proposition 5 to find the pair (N, m) with $m < a_{N+1}$ corresponding to z and the slope θ' of the derived sequence $\mathbf{d}_{\mathbf{u}}(z)$. As the pair $(1, 0)$ corresponds to $z = \mathtt{b}$, we need to analyse all pairs $(N, 0)$ and $(N, 1)$ for $N \ge 2$ and the pairs $(1, 1)$ and $(1, 2)$.

- The pair $(N, 0)$, $N \ge 2$: By Proposition 5, $\theta' = [0, \overline{2}] = \sqrt{2} - 1$.
  Let k, ℓ satisfy Items 1–3 from Lemma 18. Then $k = 6k'$ and $\ell = 6\ell'$ for some $\ell', k' \in \mathbb{N}$, $k' + \ell' \ge 1$. Dividing all parts of condition 3 by 6, we get

$$(\sqrt{2} - 1)k' - \frac{\sqrt{2}}{6} < \ell' < (\sqrt{2} - 1)k' + \frac{\sqrt{2}}{6}\,. \tag{13}$$

  We successively obtain $\ell' \ge 1$ from the left inequality and $k' \ge 2$ from the right inequality of (13). By Proposition 5, $|w| = |\pi(w)| = |r| + |s| - 2$, $|r| = p_N + q_N$, and $|s| = p_{N-1} + q_{N-1}$. As (1) implies $|r| = a_N|s| + p_{N-2} + q_{N-2} > a_N|s| = 2|s|$, we have

$$\frac{|w|}{|v|} = \frac{|r| + |s| - 2}{6k'|r| + 6\ell'|s|} \le \frac{|r| + |s| - 2}{12|r| + 6|s|} \le \frac{|r| + |s| - 2}{10|r| + 10|s|} < \frac{1}{10}\,.$$

- The pair $(N, 1)$, $N \ge 2$: By Proposition 5, $\theta' = [0, 1, \overline{2}] = \frac{1}{\sqrt{2}}$. Similarly to (13), the pair k', ℓ' satisfies

$$\frac{1}{\sqrt{2}}k' - \frac{1 + \sqrt{2}}{6\sqrt{2}} < \ell' < \frac{1}{\sqrt{2}}k' + \frac{1 + \sqrt{2}}{6\sqrt{2}}\,.$$

  From the left inequality, $\ell' \ne 0$ and, moreover, $\ell' = 1$ implies $k' \le 1$, contradicting the right inequality. Hence $\ell' \ge 2$; now $k' \ge 3$ from the right inequality. Therefore,

$$\frac{|w|}{|v|} = \frac{|r| + |s| - 2}{6k'|r| + 6\ell'|s|} \le \frac{|r| + |s| - 2}{18|r| + 12|s|} < \frac{1}{10}.$$

- The pair $(1,1)$: From (1) and Proposition 5 we have $|r| = p_1 + q_1 = 2$, $|s| = p_1 + q_1 + p_0 + q_0 = 3$ and $\theta' = [0, \overline{2}] = \sqrt{2} - 1$. Since θ' is the same as in the case $(N, 0)$, $N \geq 2$, we have the same estimate
$$\frac{|w|}{|v|} \leq \frac{|r| + |s| - 2}{12|r| + 6|s|} = \frac{1}{14} < \frac{1}{10}.$$
- The pair $(1,2)$: By (1) and Proposition 5, $|r| = p_1 + q_1 = 2$, $|s| = 2(p_1 + q_1) + p_0 + q_0 = 5$ and $\theta' = [0, 1, \overline{2}] = \frac{1}{\sqrt{2}}$. Since θ' is the same as in the case $(N, 1)$, $N \geq 2$, we have the same estimate
$$\frac{|w|}{|v|} \leq \frac{|r| + |s| - 2}{18|r| + 12|s|} = \frac{5}{96} < \frac{1}{10}. \quad \square$$

Proof of Theorem 23. By Theorem 1, it is sufficient to show that $\frac{|w|}{|v|} \leq \frac{1}{10}$ for every bispecial factor w of $\mathbf{x}_{12}$ and every return word v to w in $\mathbf{x}_{12}$. If $\pi(w)$ contains both $\mathtt{a}$ and $\mathtt{b}$, the required inequality follows from Proposition 24. If $\pi(w)$ contains only one of these letters, the inequality follows from Proposition 16. $\square$

5. Conclusion and open problems

We have shown that for balanced sequences over a d-ary alphabet, the critical exponent is greater than or equal to $\frac{d-1}{d-2}$ for $d \geq 11$. In fact, the proved result is a bit stronger: a balanced d-ary sequence contains a factor of exponent at least $\frac{d-1}{d-2}$ (thus in the case $\mathrm{E}(\mathbf{v}) = \frac{d-1}{d-2}$, the supremum is reached). Further, we have proved this lower bound sharp for all alphabets of even size $d \geq 12$ by presenting an explicit construction of balanced sequences with the required critical exponents. Based on these results, we state a new conjecture, replacing the conjecture from [26], which fails for $d \geq 11$.

Conjecture 25. *The minimal critical exponent of a d-ary balanced sequence with $d \geq 11$ equals $\frac{d-1}{d-2}$.*

This conjecture remains open for alphabets of odd size. As the next step to set the conjecture, we have found, with the aid of computer search, a balanced sequence $\mathbf{x}_{11}$ over an 11-letter alphabet with the required critical exponent $\frac{11}{10}$. The construction is very asymmetric and hard to be found by hand: $\mathbf{x}_{11} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}')$, where $\theta = [0, 5, 1, \overline{1, 1, 1, 2}]$ and the two constant gap sequences are $\mathbf{y} = (\mathtt{12})^\omega$ and $\mathbf{y}' =$
$$(1'2'3'4'5'6'7'8'1'9'3'2'5'4'7'6'1'8'3'9'5'2'7'4'1'6'3'8'5'9'7'2'1'4'3'6'5'8'7'9')^\omega.$$
We have calculated the critical exponent of the sequence $\mathbf{x}_{11}$ using our computer program based on the algorithm described in [12]. As the proof is a tedious version of the proofs from Section 4, we have not included it in the paper.

Besides Conjecture 25, we propose a few questions for further study of the minimal critical exponents of sequences given by certain natural restrictions.

- A sequence $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$ is *k-balanced* if $\big||u|_a - |v|_a\big| \leq k$ for any of its factors u and v of equal length and any letter a. Thus 1-balanced sequences are exactly balanced sequences; the Thue-Morse sequence, having the minimal critical exponent among binary sequences, is 2-balanced.
  Q1 What is the minimal critical exponent of a d-ary 2-balanced sequence?
- A sequence $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$ is *symmetric* if for any factor u and any bijection $\tau : \mathcal{A} \to \mathcal{A}$, $\mathbf{u}$ has the factor $\tau(u)$. The Thue–Morse sequence, having the minimal critical exponent among binary sequences, and the Arshon sequence [1], having the minimal critical exponent among ternary sequences, are symmetric.
  Q2 What is the minimal critical exponent of a d-ary symmetric sequence?
- Two words are *Abelian equivalent* if they have equal Parikh vectors. Replacing equality in the notion of a power with Abelian equivalence, Abelian powers, Abelian exponents and Abelian critical exponents are defined. There exist sequences with Abelian critical exponent arbitrarily close to 1 [8], but for no alphabet size d the minimal Abelian critical exponent of a d-ary sequence is known; see [28,25] for the best known lower bounds.
  Q3 What is the minimal Abelian critical exponent of a d-ary balanced sequence?

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgement

The second author was supported by Czech Technical University in Prague, through the project SGS20/183/OHK4/3T/14. The first and the third authors were supported by The Ministry of Education, Youth and Sports of the Czech Republic through the project CZ.02.1.01/0.0/0.0/16_019/0000778. The fourth author acknowledges the support by the Ministry of Science and Higher Education of the Russian Federation (Ural Mathematical Center project No. 075-02-2021-1387) and by Ural Mathematical Center, project No. 075-02-2022-877.

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